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ABSTRACT

The main purpose of this book is to provide background material in geometry for teachers or prospective teachers who know little or no geometry. It should be suitable as a text for a one-semester course for teachers of junior high school or upper elementary school students. Chapters contain developmental material and exercises. Chapters include: (1) Introduction; (2) Sets; (3) Logic and Geometry; (4) Abstractions and Representations; (5) Non-Metric Geometry; (6) Measurement; (7) Accuracy and Precision; (8) Congruence; (9) Parallels and Metric Properties of Triangles; (10) Areas, Volumes, and the Theorem of Pythagoras; (11) Circles, Cylinders, and Cones; (12) The Coordinate Plane and Graphs; (13) The Sphere; and (14) Non-Metric Polyhedrons. (RH)

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**SCHOOL
MATHEMATICS
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STUDIES IN MATHEMATICS
VOLUME V

Concepts of Informal Geometry

(preliminary edition)

By RICHARD D. ANDERSON, Louisiana State University

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Written for the SCHOOL MATHEMATICS STUDY GROUP
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PREFACE

The main purpose of this book is to provide background material in geometry for teachers or prospective teachers who know little or no geometry. It is designed for use in courses and in-service type training programs for teachers at the junior high school or upper elementary level. It should be suitable as a text for a one-semester freshman college course for prospective teachers at such level. This book is not designed to train teachers to handle SMSG tenth grade geometry but it might be used for background information and points-of-view. Volume II of this series is designed with the tenth grade course in mind.

If this text is used for in-service programs for upper elementary teachers, then some selectivity of subject matter would be called for. Chapters 1-8 probably should be used with some sections of Chapters 7 and 8 taken lightly. The "proofs" in Chapters 9 and 10 might be omitted. The elementary portions of Chapters 11, 12, and 13 might well be used. Chapter 14 is primarily intended for junior high school teachers who will be using SMSG materials.

There will be considerable review of geometric ideas but the review will be phrased partly in terms of present day "set" language. Where possible and appropriate, both traditional language and set language will be used to clarify each other.

It is not intended that this book give a complete review or cover all details mentioned in the experimental SMSG junior high school texts. It is intended that this book stress basic understandings of ideas, concepts and points-of-view. In particular, emphasis is put on the interrelationships between the concepts of and use of measurement, congruence, the real number system, and various geometric systems. The author hopes that the broad outlines of "good" mathematical developments will come through.

Clear-cut definitions and explicit assumptions are made where increased understanding will result. But the author has tried to keep in mind that this is not a treatise on abstract geometry. The intuitive and informal approach is emphasized throughout.

One body of material that has been omitted from this book is that dealing with sets of concurrent lines associated with triangles: medians, angle bisectors, etc. Some people teaching from this volume may want to use such material for special projects or the like.

In studying this material, one should have a pencil and paper handy and be prepared to draw figures to help understand the developments. The reading of mathematics is not like the reading of novels. One may have to read the material several times to understand it. Some prefer a "light" reading of a section or chapter to get general ideas before detailed study.

The author and SMSG will appreciate suggestions regarding the suitability or non-suitability of this volume for the purposes suggested above. It is the intention that this volume be later reproduced in revised form. Suggestions concerning the revision are welcomed and should be sent to

School Mathematics Study Group
Drawer 2502A, Yale Station
New Haven, Connecticut.

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Chapter 1

Introduction

Geometry is concerned with the study of spatial relationships. This study, of course, includes what is usually called "plane geometry" for a plane (a flat surface) is regarded as a part of space. Traditional tenth grade geometry is more than simply a study of spatial or planar relationships; it is the setting for the development of a mathematical logical or axiomatic system.

In the SMSG materials, the geometry which is found in the Junior High School texts is intuitive geometry, the development of geometric (spatial) points of view and thought and the understanding of spatial relationships. It is not axiomatic as such. Questions of informal deduction naturally arise and where appropriate are dealt with by informal arguments.

In the past, geometry has been a vehicle for teaching accuracy of language, expression and thought. To some extent the 7th and 8th grade geometry is dedicated to this end. In particular, set language simplifies mathematical vocabulary and at the same time forces both considerable precision of expression and emphasis on the meanings of concepts. Traditional Euclidean geometry went part way in this direction. The SMSG materials (both Junior High and

10th grade) go considerably farther in making clear cut definitions and in making some distinctions which were only implicit in Euclid.

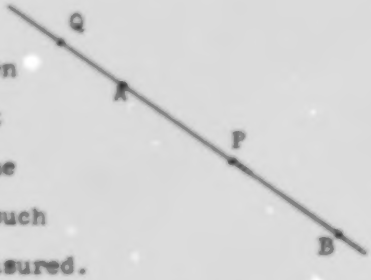
The consistent use of set language in geometry has three other important values to the student. First, the set point-of-view is of fundamental importance in much of present day mathematics and an appreciation of it helps produce a certain amount of mathematical maturity. Second, set language itself gives students a unifying thread which runs through much of their mathematical studies. No longer will it be true that students view 9th grade algebra and 10th grade geometry as essentially unrelated subjects. Third, use of set language actually should make many ideas of mathematics substantially easier to grasp for the student. Set language simplifies rather than complicates. It frequently forces attention on the proper concept.

It should be pointed out, however, that the set point-of-view is no panacea by itself. Mathematics will remain a very substantial subject. Furthermore, it is not proposed by SMSG (or almost anybody else) that set theory as such be taught to high school students. It is proposed that the language of sets be used. The language of sets is rather straightforward and simple--once you get on to it--but the subject of set theory gets deep and delicate rather quickly. Set theory itself should probably be left to professional mathematicians or to those who are thinking seriously of becoming such.

Let us illustrate some of the ambiguity in traditional terminology and notation and our attempts at eliminating it.

One of the often remembered properties of Euclidean geometry is that "a straight line is the shortest distance between two points". Now, really, there are at least three different concepts which are confused in this statement. We discuss these concepts.

(1) A straight line is usually thought of as a set of points (the set or collection of points on it). For any two points there is exactly one straight line containing them (i.e., two distinct points determine a straight line).



The straight line containing A and B contains some points (like P) between A and B and some points (like Q) not between A and B. We shall denote the line AB by $\overleftrightarrow{AB}$. A straight line as such does not have length--it can not be measured.

(2) A segment is a part of a straight line. In particular, the segment AB (denoted $\overline{AB}$) is the set of points consisting of A and B and all points between A and B. A segment has length --it can be measured. The length of $\overline{AB}$ is denoted by AB or sometimes by $m(\overline{AB})$.

(3) A distance is a number (or a number of units). In geometry, for any two points there is a distance between them --the distance being the length of the segment joining them.

Whatever is customarily meant by straight line--and geometry books are vague on this--a straight line is not a distance.

The statement "A straight line is the shortest distance between two points", then, confuses the concepts of straight line, segment, and distance. However, the statement does communicate something of what is intended. But simpler and more precise language would make for greater clarity. We could say "A segment is the shortest path between two points". This statement is an improvement on the earlier. It would be better, however, if we had defined or explained the meaning of the word "path". In Chapter 9, the "triangle inequality" property does clarify the meaning of the "shortest distance" statement. In other chapters of the book, considerations of the type suggested here will be greatly amplified.

Having made the observation that terms used in mathematics should have explicit and clear-cut meanings, we agree that we cannot achieve perfection in this respect. In particular, there are several terms which are consistently used with dual meanings but for which the particular meaning intended is almost always clear. Examples of such terms are "radius" of a circle which means a number (usually) but sometimes a set of points, "side" of a triangle or polygon which means either a number or a set of points and "base" and "altitude" of a figure which also have similar dual meanings. These words are so widely used and well understood that it seems inadvisable to insist on one meaning or the other.

Chapter 2

Sets

1. Terminology.

One of the important ideas of mathematics is that of "set." Synonyms for the word "set" are "collection", "family", and "aggregate." The term "set" is used in mathematics in much the same sense as it is occasionally used in ordinary language. In geometry we speak of a line as a set of points. Or we may speak of the set of all lines which contain a given point. In arithmetic, we speak of the set of all positive even whole numbers, that is, 2, 4, 6, etc.

In everyday language, we talk about the set (or collection) of books in the city library, the set of pupils in the seventh grade of a school, or the set of all red-headed children less than two years of age.

In order for a set to be defined or understood, there must be some clear-cut criterion for deciding whether any particular object is in the set or is not in the set. We speak of the objects in a given set as the "elements" or "members" of the set. For instance, consider the set of pupils in the seventh grade of West Junior High School. An object is an element of this set if (and only if) the object is registered as a seventh grade student in West Junior High School. Therefore, we can tell whether a given object is in the set.

Notation. It is useful to let symbols denote sets. We frequently use capital letters for this purpose. Thus, when convenient, we may let "A" be the set of all positive even whole numbers, or "M" be the set of all grade school children who can swim. Braces, { }, are frequently used in describing sets. Thus B = {Mary, James, William} describes the set B whose elements are Mary, James and William. Or C = {1, 3, 5, 7, 9} describes the set of odd counting numbers less than ten. Note that in each of these latter cases we have actually enumerated the elements of the set B or C. We use three dots to suggest "and so on". For example, the set A of positive even whole numbers is sometimes written A = {2, 4, 6, ... }.

In set notation as in other mathematics we use the symbol "=" (equals or is equal to) to mean "is the same as." Note that above, B and {Mary, James, William} are different names for the same set.

Subsets. Let Y be the set of states of the United States which contain cities east of the Mississippi. Let Z be the states which were the original 13 states. Then every element of Z is an element of Y. We say that Z is a subset of Y (or Z is contained in Y). We may write $Z \subset Y$ and we read it "Z is contained in Y." We may also say Y contains Z, or $Y \supset Z$. Notice that the open part of the symbol $\subset$ or $\supset$ is toward the set which contains the other as a subset.

In general, the set R *is a subset of the set* T *if each element of* R *is an element of* T. We may observe that each set is a subset of itself; in notation, if X is any set, $X \subset X$ (X is contained in X).

Let U be the set of all classrooms in your school. Let V be the set of all classrooms in your school with women teachers. Then $V \subset U$, i.e., each element of V is an element of U. If your school has no men teachers, then also $U \subset V$, i.e., each element of U is an element of V. In this case, $V = U$. In general, we can say that if A is a set and B is a set, and if $A \subset B$ and $B \subset A$, then $A = B$, i.e., A and B are simply different names for the same set.

Intersections of Sets. Let G be the set of all girls who are pupils in your school. Let R be the set of all red-headed people in the world. Let W be the set of all red-headed girls in your school. Then W is a subset of R and also of G. In fact, W consists exactly of those elements which are in R and are also in G. We speak of the set W as the *intersection* of the sets R and G and, in notation, we write $W = R \cap G$. The symbol "$\cap$" is called the intersection symbol. We read $R \cap G$ as "R intersection G" or "the intersection of R and G."

Let A be the set of all positive whole numbers. Let B be the set of all real numbers less than 8. Then $A \cap B$ is the set of all numbers which are in A and are also in B. In other words, $A \cap B$ is the set of all objects which are

(1) positive whole numbers and

(2) numbers less than 8.

Clearly then

$$A \cap B = \{1, 2, 3, 4, 5, 6, 7\}.$$

Definition. If X and Y are sets, then the *intersection* of X and Y (in notation $X \cap Y$) is the set of all elements each of which is an element of X *and* is an element of Y.

To determine whether an object is in $X \cap Y$ is simple: the object must be in X and must also be in Y.

Exercises 2-1

Where appropriate, use brace notation to write out your answers.

1. Let X be the set of letters of the alphabet which precede g. Let Y be the set of vowels which precede v.

   (a) $X = \{\ ?\ \}$.

   (b) $Y = \{\ ?\ \}$.

   (c) $X \cap Y = \{\ ?\ \}$.

2. Let H be the set of types (sizes) of silver coins in circulation in the United States. Let K be the set of types of coins in circulation in the United States.

   (a) Is $H \subset K$?

   (b) Is $K \subset H$?

   (c) What is $H \cap K$?

3. Let P = [3, 5, 7, 11, 13, 17, 19]

   Let Q = [1, 4, 7, 10, 13, 16, 19]

   (a) $P \cap Q$ = [?]

   (b) Is $P \cap Q$ a subset of Q?

4. Let V be the set of positive odd whole numbers. Let W be the set of positive whole numbers less than 20. Let X be the set of whole numbers divisible by 5.

   (a) $V \cap W$ = [?]

   (b) $W \cap X$ = [?]

   (c) $(V \cap W) \cap X$ = [?]

   Note that $V \cap W$ is itself a set and the intersection of this set with X is what is meant by $(V \cap W) \cap X$.

5. Let A be the set of men who have been President of the United States at some time since 1922. Let B be the set of men who have been Vice-President some time since 1922.

   (a) A = [?]

   (b) $A \cap B$ = [?]

   (c) Show that B is not contained in A; i.e., exhibit an element of B which is not an element of A.

6. Let M be the set of positive whole numbers.

   Let H be the set of multiples of 5.

   Let K be the set of multiples of 3.

   (a) H = [?]

   (b) K = [?]

   (c) $M \cap K$ = [?]

(d) Show that H is not contained in K.

(e) $H \cap K = \{ \ ? \ \}$

(f) Is $(H \cap M)$ M?

2. Union of Sets.

Let $A = \{1, 5, 9\}$ and let $B = \{2, 3, 4, 5\}$. The intersection of A and B (i.e., $A \cap B$) is the set $\{5\}$ consisting of the single element 5. How are we going to refer to the set $\{1, 2, 3, 4, 5, 9\}$? In other words, what will we call the set whose elements are the elements of A together with the elements of B? We use the word "union" in this sense. It suggests the combining or uniting of the sets. Thus $\{1, 2, 3, 4, 5, 9\}$ is the union of A and B. In notation, we write $A \cup B$ (the "union of A and B" or "A union B"). Similar notation and terminology is used for any pair of sets. Let X and Y be any sets at all. Then $X \cup Y$ (the union of X and Y) is the set consisting of the elements of X together with the elements of Y.

To determine whether or not an object is an element of $X \cup Y$ is simple. The object is in $X \cup Y$ provided it is in X or it is in Y. It could be in both.

Let M be the set of people in your school with last name "Smith." Let N be the set of people in your school with first name "John." Then $M \cup N$ is the set of all people in your school who qualify on either of two counts: for a person to be in $M \cup N$, either his last name must be Smith or his first name must be John; i.e., either the person is in M or he is in N. (Any person named John Smith qualifies on both counts.)

Empty Set. What is the set $M \cap N$ (the intersection of M and N)? To be in $M \cap N$, a person in your school must have last name Smith and first name John. Thus $M \cap N$ is the set of "John Smiths" in your school. Now suppose your school doesn't have anybody in it named John Smith. Then the set $M \cap N$ (M intersection N) doesn't have any elements in it. In this case, we say that $M \cap N$ is the empty set (or null set). Some people would claim that $M \cap N$ isn't a set if it doesn't contain any elements. But mathematicians generally find it more convenient and useful to use the concept of the empty set. Then if X and Y are sets, $X \cap Y$ is a set. And $X \cap Y$ is empty if and only if no element of X is an element of Y. The empty set, thus, is the set with no elements in it. We use the symbol $\emptyset$ to denote the empty set.

Sometimes in describing a set, we may not know, at first glance, whether or not the set has any elements in it. If the set contains no elements then we are simply describing the empty set. For example, if your school has nobody in it named Smith, then M would be the empty set.

Exercises 2-2

Use brace notation where possible and appropriate.

1. Let $A = \{1, 3, 5, 7, 9\}$

   $B = \{1, 2, 3, 4, 5\}$

   $C = \{2, 4, 6, 8, 10\}$

   Find:

   (a) $A \cup B$

   (b) $A \cap B$

   (c) $A \cup C$

   (d) $A \cap C$

2. Let X be the set of states of the United States whose names begin with a direction (e.g., West Virginia). Let Y be the set of states which border on the Pacific Ocean.

   Find:

   (a) X

   (b) Y

   (c) $X \cup Y$

   (d) $X \cap Y$

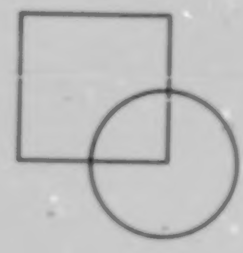
3. Let M be the set of points on or inside the square. Let N be the set of points on or inside the circle. Draw similar figures and shade

   (a) M

   (b) N

   (c) $M \cup N$

   (d) $M \cap N$



4. Describe two sets H and K such that

   (a) $H \cap K$ is empty, and

   (b) $H \cup$ [illegible] not empty.

5. If A and B are sets, $A \cap B$ is empty and $(A \cup B) \subset A$, what can you conclude about B?

6. If M is a set and N is a set and if $(M \cap N) = (M \cup N)$, what can you conclude?

7. Let R be the set of all positive even whole numbers. Let S be the set of all positive whole numbers divisible by 3.

   (a) Describe $R \cap S$

   (b) List three positive whole numbers not in $R \cup S$.

8. Explain why for any sets X and Y, $(X \cup Y) \supset (X \cap Y)$.

3. One-to-one correspondences.

Let α (the Greek letter alpha) be the set of capital letters in the English alphabet. Let β (the Greek letter beta) be the set of lower case letters.

$$\alpha = \{A, B, C, \ldots Y, Z\}$$

$$\beta = \{a, b, c, \ldots y, z\}$$

Now there is a natural way of associating the elements of α with the elements of β so that each element of α corresponds to exactly one element of β and each element of β to exactly one element of α under this same association.

2.10

Each capital letter is to correspond to its lower case letter. We use the symbol ↔ or ↕, as appropriate, to indicate the matching or correspondence. Thus,

$$\begin{array}{cccc} A & B & ----- & Z \\ \updownarrow & \updownarrow & ----- & \updownarrow \\ a & b & ----- & z \end{array}$$

This is an example of a "one-to-one" correspondence between the sets α and β. There are other one-to-one correspondences between α and β. Thus we might let Z correspond to "a" and each other capital letter correspond to the lower case letter following it. Thus

$$\begin{array}{ccccc} A & B & ------ & Y & Z \\ \updownarrow & \updownarrow & ----- & \updownarrow & \updownarrow \\ b & c & ----- & z & a \end{array}$$

In many cases in life, we are interested in two sets and the existence or non-existence of a one-to-one correspondence between the two sets. In some instances, we are interested in a particular matching process (one-to-one correspondence), not just any one. If you are giving a theater party for 10 boys and 10 girls, your set of tickets should be in one-to-one correspondence with the set of people going. If the seats are reserved it probably makes a great deal of difference what one-to-one correspondence you set up as you pass out the tickets to the various members of the party.

A one-to-one correspondence between two sets M and N, then, is a matching of the elements of M with the elements of N so that under this matching each element of either set corresponds to a particular element of the other (which in turn corresponds to it). No element of either set can be left over.

In most homes, there is a one-to-one correspondence between the set of chairs at the dinner table and the set of members of the family. Furthermore, we are especially aware of that particular one-to-one correspondence which matches each person with his own chair.

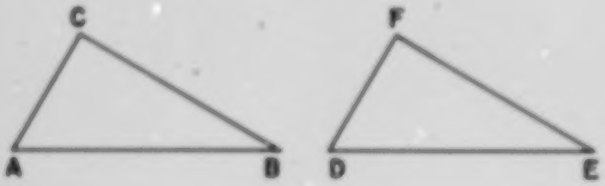
One-to-one correspondences are of fundamental importance in the process of counting. A person learns to count--meaningfully--when he learns to match the counting numbers in order and up to a certain number with the objects he is trying to count. The process of counting is a process of establishing a one-to-one correspondence. Even before children learn to count, they are frequently aware of one-to-one correspondences. Take four small boys and three ice cream cones. Even before the cones are passed out, some boy may well have mentally matched the set of boys with the set of cones and anticipated certain difficulties.

In geometry the notion of one-to-one correspondence arises naturally and significantly. Consider two congruent triangles as below.

Let A↔D

B↔E

C↔F



Under this correspondence of the set {A, B, C} of vertices of the left triangle with the set {D, E, F} of vertices of the right triangle the two triangles seem to be congruent.

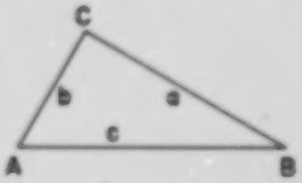
But under the correspondence

$$\begin{array}{c} A \longleftrightarrow D \\ B \longleftrightarrow F \\ C \longleftrightarrow E \end{array}$$

the triangles do not seem congruent for the side AB is not the same length as the side DF.

Exercises 2-3

1. Is there a one-to-one correspondence between the states of the United States and cities (in the United States) of over 1,000,000 in population? Why?
2. Consider the triangle in the figure. List all six possible one-to-one correspondences between the set of vertices {A, B, C} and the set of sides {a, b, c}.



3. If set R is in one-to-one correspondence with set S and set S with set T, is there a one-to-one correspondence between set R and set T? Explain.

4. Describe three different one-to-one correspondences between the set of digits {1, 3, 5, 7, 9} and the set of symbols $\{\cap, \cup, \supset, +, \div\}$.

5. Describe a one-to-one correspondence between the set of positive integers and the set of negative integers.

Chapter 3

Logic and Geometry

1. Statements and Implications of Statements.

When we write a sentence, we make a statement. The statement may be true or it may be false or it may be meaningless. Examples of meaningless statements are:

(1) Abadab diaha loween syman.

(2) Horses and chairs ride honor among windows.

In (1) the "words" don't even make sense. In (2), while the words all make sense the sentence itself does not; (2) is in the form of a sentence but it does not have meaning. For the purposes of the discussion of this chapter we want to consider statements that are not meaningless.

So we restrict our attention to meaningful statements. There is another distinction we would like to make. When one makes a statement, he is trying to communicate information (valid or invalid). Many statements that are made in everyday language are true in spirit but false as actually stated. They communicate a valid idea but are not technically correct. For many purposes technical correctness is not especially important.

But in subjects like mathematics we have to be concerned with the correctness or non-correctness of the specific statements we make. It is in the nature of mathematics that precision of language and thought is important. Therefore it is necessary for us to study the significance of statements, their meanings and their implications. We shall assume that statements mean what they say and not merely what we might wish them to say. However, statements are usually made in conjunction with other statements and also on the basis of tacit agreements which have been built up in general or in the particular discussion. In Chapter 7 we discuss this aspect of language further. Here we cite an example. The statement, "I am not going to eat breakfast," usually carries with it a tacit time understanding. A person who made this statement on getting up in the morning and then ate breakfast that morning would be considered as having made an untrue statement. Furthermore, if he did not eat breakfast that morning, but did the following morning, his original statement would be considered to be correct. It would normally have been understood that he was referring to breakfast the day he made the statement unless the contrary was specified. Thus we agree that individual statements should be understood to be in context, more to restrict or clarify their meanings than to "change" them.

It is convenient to let symbols like A, B, and C denote statements. For instance, consider A to be the statement, "The weather is not clear today," and consider B to be the statement, "I am going to stay home." We can make (further) statements using statements A and B as "building blocks".

Example 1. A is true. In our illustration this says "The statement 'The weather is not clear today,' is true." But this latter assertion means nothing more nor less than the original statement "The weather is not clear today." Either statement is true provided the other is. Thus we conclude that "A" and "A is true" really mean the same thing.

Example 2. B is not true. In our illustration this says "The statement 'I am going to stay home' is not true" or in other words "I am not going to stay home." The statement "B is not true" is called the negative of B and can frequently be achieved by the insertion of the word "not" in the proper place in the statement B.

Example 3. "A and B" (or what is the same, "A is true and B is true"). In order for statement "A and B" to be true, both A and B individually must be true.

Example 4. "A or B". The statement "A or B" will be true provided at least one of the two separate statements "A" and "B" is true. In other words, "A or B" is true unless both "A" and "B" are false. The statement "The weather is not clear today or I am going to stay home" is true unless (i) the weather is clear today and (ii) I do not stay home. (The statement "A or B" has meaning but in our illustration, it is not the kind that is made in ordinary speech, as the statements A and B themselves are not "natural" alternatives.)

Example 5. "If A, then B." In our illustration, "If the weather is not clear today, then I am going to stay home." This is known as a statement of implication. Another way of making this statement is to say "A implies B". The statement means that it cannot be that A is true and B is false. The statement says nothing about B in the event A is not true. Consider our illustration. In the event the weather is clear today, I am at liberty to stay home or not as I see fit. The original statement of implication does not restrict my behavior if the weather is clear. In the event A is not true, the statement "If A, then B" has meaning and is certainly not false. Thus, in this event, we must consider the statement of implication to be true even though it does not contribute information about B.

The Contrapositive. Statements of implication (If A, then B) are of great importance in mathematics. They are widely used. "If x is divisible by 4, then x is divisible by 2." "If corresponding sides of two triangles are congruent, then the two triangles are congruent." Any statement of implication can be made in a variety of ways. We have already noted in Example 5, that "A implies B" means "If A, then B." The statement "If B is false, then A is false" is called the contrapositive of the statement "If A, then B." A statement of implication and its contrapositive really mean the same thing. We can see this by considering the following table. In this table we have listed four statements across the top: "A", "B", "A implies B", and "B is false implies A is false." In the left two columns we have listed the four possibilities for statements A and B. The bottom row, for instance, lists A as false and B as false.

A	B	If A, then B	If B is false, then A is false
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

In the third and fourth columns are listed T and F according as the statement at the head of the particular column is true or false for A and B as listed in the same row. Thus the statement "If A, then B" is shown as false for A "true" and B "false". So also is its contrapositive as listed at the head of the fourth column. If B is false, then A cannot be true. Because the third and fourth columns are alike, we conclude that the statement "If A, then B" and its contrapositive have the same meaning. If either is true the other is. If either is false, the other is. The contrapositive is important, in part, because some statements of implication are easier to recognize as true (or false) when stated in the form of the contrapositive.

Equivalent Statements. A statement of implication and its contrapositive are examples of equivalent statements. So are the statements "A implies B" and "If A, then B". In general, two statements, P and Q, are said to be equivalent if P implies Q and Q implies P. In other words, if either statement is true, the other must also be true. Looking at this informally we may say that P and Q are equivalent if they are different ways of saying the same thing. Let us give an example. Suppose M and N are sets.

Let P be the statement: M is a subset of N.

Let Q be the statement: Each element of M is an element of N. Then P and Q are equivalent for

(1) If P is true, then Q is true, and

(2) If Q is true, then P is true.

Or we can say, (1) P implies Q and (2) Q implies P. We might note that equivalence has the following property, "If each of two statements is equivalent to a third, then they are equivalent to each other."

The Converse. A statement of implication has a converse, which, in general, is not equivalent to the statement. The converse of the statement "A implies B" is the statement "B implies A". Clearly if the statement "A implies B" and its converse are both true then A is equivalent to B. The converse is particularly important in geometry. We make a statement in the form "If A, then B". We are frequently also interested in the statement "If B, then A".

Consider the following valid proposition of geometry: "If two angles are vertical to each other, then they are congruent to each other." This is sometimes stated in the form "Vertical angles are congruent." The converse of this statement would be: "If two angles are congruent to each other, then they are vertical to each other." This converse is not a valid proposition of geometry (i.e., is not true) for we may exhibit two angles which are congruent to each other but which are not vertical to each other.

Exercises 3-1

1. Let P be the statement "6 is an even number," and let Q be the statement "all whole numbers between 5 and 9 are even". Write out the statement indicated (whether or not such is true).
   (a) P and Q
   (b) P or Q
   (c) If P, then Q
   (d) Q is not true (be careful how you do this)
   (e) If Q is not true, then P is true.
2. In each of (a) through (e) of 1. state whether the statement given is true.
3. Explain why it is true that if each of two statements is equivalent to a third, then the two statements are equivalent to each other.

4. Suppose x and y are numbers. Consider the statement of implication: If $x \cdot y$ is positive, then x is positive.
    (a) State its converse.
    (b) State its contrapositive.
    (c) State which of (a) and (b), if either, is a true statement.
5. Give an example of your own of a statement of implication
    (a) which is true.
    (b) which is false.
    (c) which is true but whose converse is false.
    (d) whose converse is true.
    (e) whose contrapositive is true.
6. If you are at least vaguely familiar with the notions of congruence and vertical angles, draw two congruent angles which are not vertical thus justifying the last statement preceding the exercises of this section.

2. Postulates and Proof.

In any discussion, we assume a good many things. We assume that specific words mean what we understand them to mean. We assume the properties of elementary logic--that sentences mean what they are supposed to; for example, that the statement "If A is true, then B is true" is equivalent to its contrapositive: the statement, "If B is not true, then A is not true". We also have to assume some properties of the particular subject matter under discussion. In Euclidean geometry, for instance, we usually assume that a line is a set of points and that for any two points there is exactly one (straight) line containing the two points.

The assumptions we make are, so to speak, a point of departure for our further study. In formal geometry, we usually call the assumptions "postulates". And we try to write down specifically what we are assuming to be true. Otherwise we would have a rather fuzzy base of operations. On the basis of our assumptions we can then draw certain conclusions by use of elementary logic. We sometimes call conclusions we can draw "theorems" or "propositions". The justifications for the various conclusions are called proofs. A proof of a theorem is an explanation of why the statement of the theorem must be true (or cannot be false).

It is necessary to make definitions of words we use if the meanings are not already clearly and unambiguously understood. Thus words like "angle", "triangle", and "circle" should be defined in geometry. Words like "and", "is", "there", and "or", are considered to be understood. There are some words for which we do not or cannot give explicit definitions. These will be the so-called undefined terms of our system. In geometry, "point" "line" and "plane" are examples of undefined terms or concepts. The postulates tell us what we assume to be true about points, lines, and planes. The theorems tell us what we can conclude to be true.

Geometry, like other mathematical subjects, is not just a formal system of definitions, postulates, theorems and proofs to be studied, learned, memorized, and (if possible) understood. The development of intuition and the understanding of ideas is at least as important as the "proof" side of geometry. Geometry in the junior high school is particularly concerned with introduction of terminology, the understanding of spatial concepts, and the development of more geometric intuition. Understanding refers to comprehension of ideas and language. It involves learning of facts together with interrelationships of these facts. It is not simple memorization. Intuition refers to the anticipation of facts and ideas before these are pointed out by others. A person with good geometric intuition can frequently decide for himself what the facts are and what the theorems ought to be. Naturally, at the junior high school level, only a small amount of this type of intuition can be expected.

While proofs as such are not stressed in this book, some explanation of the form and methods of proof is called for. Let us consider an example. Suppose we have statements A and B and we wish to prove that A implies B, i.e., that the statement "If A, then B" is true. We call "A" the hypothesis and "B" the conclusion of the statement.

Sometimes B as a statement is simply a rewording of A (or is immediately implied by A) in which case the proof might occasionally properly be stated as "obvious".

More often, however, the statement "If A, then B" is not immediately obviously true. One possible method of proof is to find intermediate steps in a "direct" type argument. Perhaps we can find statements C and D such that

A implies C,
C implies D,
and D implies B.

Then we may conclude that A implies B. For, note that if A is true, then C must be true, which means that D must be true, which means that B must be true (which is what we wanted to show).

When the proof is in the form of a sequence of statements like the above, it may be that each step can be justified by one known property. If so, the argument is usually easy to follow.

But it may be that each step needs a fairly lengthy proof itself. In such cases the form of the argument may get complicated. But the idea of the argument may still be simple.

Another method of argument is the so-called "indirect method" or argument by contradiction. Suppose we want to show that "If A is true, then B is true". If this statement were false, then

(1) A would be true and
(2) B would be false.

We suppose both of these are so. Specifically we suppose B to be false. If as a consequence of A being true and B being false it follows that some (other) statement is both true and false, then we have a contradiction, i.e., a situation that cannot logically arise. Hence our assumptions cannot all be true. Therefore it cannot be that A is true and B is false. Hence if A is true then B must also be true which was what we wanted to show.

Examples of indirect arguments are scattered throughout the book. We give an elementary example of such an argument here. We regard a straight line as a set of points. Suppose we have given the property that for any two distinct points, there can be at most one straight line containing them. We wish to prove "If two distinct straight lines intersect, then their intersection cannot contain two distinct points". The proposition is of the form "If A, then B". We suppose B to be false; i.e., we suppose the intersection does contain two distinct points. Then

(1) each of the two distinct straight lines of our hypothesis does contain the two distinct points.

(2) at most one straight line can contain the two points (as is known from the given property).

Statements (1) and (2) contradict each other. We have a contradiction. Hence the statement "B is false" cannot be true (if A is true). Thus "If A is true, then B is true" as was to be shown.

3.13

In traditional 10th grade geometry, proofs are usually given in a form of

(1) statement	(1) reason
(2) statement	(2) reason
"	"
"	"
"	"
(k) statement	(k) reason
Q.E.D.	

The final statement (k) is usually the assertion of the conclusion of the theorem; i.e., "that which was to be shown" or, in Latin, "Quod Erat Demonstrandum".

In actual practice in mathematics, however, proofs are almost never given in this form. A proof is written out as a paragraph or several paragraphs. The formal presentation in geometry texts is designed not as the only way to present a proof, but rather as a means of emphasizing the significance of implication, the dependence on previously established results, and a logical step-by-step procedure. In junior high school geometry, a more casual form for a proof seems called for in those few cases where proofs, as such, are needed. The critical aspect of any proof is not its form but its validity, i.e., its logical soundness.

Finally we ask how we might show that an "alleged" theorem is false (or not valid). We might be given a "proposition" and be asked to determine whether it is true or false. If the "proposition" asserts something to be so for all cases of a certain type, then we can disprove the proposition by exhibiting an example of this type for which the assertion is not so. Consider the statement, "All primes are odd numbers." We can disprove this statement (show it false) by exhibiting the number 2 which is a prime and is not an odd number.

Exercises 3-2

1. Write out two or three of the postulates of geometry (as best you can remember them).
2. Recall (as best you can) some proposition of geometry that we haven't mentioned. Write it in the "If--then--" form. Write its contrapositive and its converse, if possible. (For some propositions these are rather tricky.)
3. Write three "theorems" about numbers (in the "If--, then--" form). Write the converse of one of these and the contrapositive of another.
4. Write out an "alleged" theorem of geometry which you can disprove by example.
5. Explain why the following two statements are not, in general, equivalent.

   (a) If A, then B .

   (b) If A is false, then B is false.

Chapter 4

Abstractions and Representations

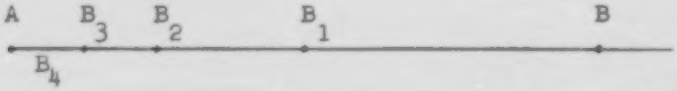
For almost all people who study any mathematics, the subject matter is properly regarded as a tool for use in problems that arise in everyday living. Some of these problems are technical or scientific in nature but most are applications of arithmetic. The problems of arithmetic usually deal with counting or with measurement or with both.

The system of numbers which we use is, however, an abstract system. There are infinitely many counting numbers, {1,2,3,...} but in applications we never count more than a rather small finite number of objects in the world about us. It turns out that assuming the existence of infinitely many counting numbers is extremely useful in mathematics whether or not the numbers can be considered to correspond to concrete objects in our universe. It is mathematically (but not physically) unimportant as to whether or not there are infinitely many objects in our universe. But the mathematics we get from the assumption of infinitely many counting numbers is of tremendous importance in the scientific world of today. There would be no modern science or technology if such basic assumptions in mathematics had not been made a long time ago.

Hence, we should be prepared to accept mathematical systems (like the number system) as abstractions of phenomena in the everyday world. Abstract mathematical systems have helped us and will continue to help us understand our environment.

The basic concepts of geometry are also mathematical abstractions. A plane, for instance, is a mathematical abstraction of a flat surface. When we want to study the common characteristics of flat surfaces we study planes. We specify properties of planes by thinking of common properties of flat surfaces like walls, floors, blackboards, etc.

For any two points (of a plane) there is a point half-way between them. This property of planes (or of lines) suggested by thinking about flat surfaces leads to a distinction between the mathematical abstraction and the physical reality.



On [illegible]e mathematical plane there must exist the point B_1 (halfway between A and B), the point B_2 (halfway between A and B_1), the point B_3 (halfway between A and B_2), and so on. The "halving the length" process can be considered continued indefinitely. In the mathematical abstraction this seems reasonable. But on any flat surface such a process could be performed only a very small number of times before the "points" would be indistinguishable. Try to think of it being performed even 50 times for instance. Even with the sharpest instruments it would not be possible.

Where do these considerations leave us? Concepts like these concerning the mathematical plane have turned out to be extremely useful in helping us understand not only mathematics itself but also many applications of mathematics. Even though the mathematical abstraction does not seem to give a "true" picture of the physical object, it frequently is of great value. A well-known example of this type of reasoning is the use of maps for the surface of the earth. The usual (flat) map of the earth (Mercator Projection) involves considerable distortions in extreme latitudes and does not correctly indicate "shortest" paths for long distances. Nevertheless, such maps are widely used and make possible better understandings of the surface of the earth. The abstractions from the surface of the earth to the surface of a sphere and from the surface of a sphere to a flat surface such as a map are important, valuable, and practical.

It is interesting to note here a difference between pure and applied mathematicians. Pure mathematicians study mathematical systems as such whereas applied mathematicians study applications of such systems to various problems that arise in the world about us. Both groups of people are important. Some of the really important scientific advances have come as results of pure mathematicians' better understanding of mathematical systems. The development of analytic or coordinate geometry (discussed in Chapter 12) was a result of pure mathematics--an attempt to understand relationships between mathematical systems. Without something like analytic geometry we probably would have no modern science.

There is another side to the coin of abstraction. While mathematical systems are abstractions of physical phenomena, we frequently study the mathematical objects by considering specific representations of them. A blackboard is a representation of a plane. A drawing of a line is a representation of the line; it is not the line. We often can understand mathematical systems better by considering concrete representations of them. In fact, much of our intuition about mathematical systems comes from considering representations of them. Our intuition about geometric space --space as a set of points--comes from our natural awareness of physical space--the three-dimensional environment in which we live. But we should not confuse the mathematical system with its representation. We may think of the walls of a room as planes. However, the walls are not the planes, just models of them.

Sometimes our language leads to confusion on this score. We should try to think, speak, and write with clarity and precision.

The statement "Draw a line" really means "Draw a representation of a line." While for simplicity we may use the expression "Draw a line" we should keep in mind what is meant by it.

Because we shall regard drawings as representations of abstract mathematical objects or entities, it really is not important mathematically how "accurate" our drawings or sketches are. Drawings and sketches are to suggest ideas. Whether we "draw a line" freehand or with a straightedge makes no difference mathematically, the thing drawn is only a representation of a line anyway. Whether we make drawings freehand or with instruments may, however, make some difference pedagogically. The nature of the audience and the uses to which a drawing is to be put will frequently determine the type of drawing to be made. We should be sufficiently careful in sketching to get our ideas across. We should not be so meticulous that the processes of drawing either interfere with the effective communication of ideas or replace mathematical concepts with artistic ones.

In classical geometry, the unmarked ruler and compass were the "tools" that were allowed. Questions concerning geometric constructions using only these "allowable tools" are legitimate ones in geometry. These questions can be (but usually are not) phrased in terms of abstract concepts and processes.

Exercises

1. In your own words describe what is meant by a mathematical system as an "abstraction".
2. Explain how the symbols used for numbers may be regarded as "names" or "representations" of the numbers.
3. Without looking ahead to the next chapter, describe or define a "triangle". Keep your definition for comparison with that of the text.
4. Without looking ahead to Chapter 5 and Chapter 6, describe or define an "angle". Does a triangle "have" any angles by your definition? Re-examine your definitions later.

Chapter 5

Non-Metric Geometry

1. When we say non-metric geometry, we are referring to that part of geometry which does not have to do with measurement. We might call it no-measurement geometry. In this chapter we shall be reviewing and restating several of the important facts and points-of-view of traditional Euclidean geometry. But in accordance with the chapter title we shall concern ourselves with that fragment of Euclidean geometry which is independent of measurement. Very little of the terminology of this chapter will not be familiar to most readers. We shall, however, give special or restricted meanings to a few of the words.

We consider space (an abstraction of ordinary every-day three-dimensional space) to be a set of points. Intuitively speaking, a point represents and is represented by a position or location in space.

We shall give some of the basic properties of space and its subsets. There are certain subsets of space which are of fundamental importance in Euclidean geometry. The most important of these are (straight) lines and planes. Each (straight) line is a set of points of space and each plane is a set of points of space. We shall understand that each line extends indefinitely far in both directions. Later, we shall specifically think of portions of lines. In geometry, we study such things as properties of the set of all lines in space or the set of all lines in a plane and we study properties of the set of all planes in space.

We intuitively understand a line to be what we think of as straight and a plane to be what we think of as a flat surface. To study flat surfaces, we abstract the notion of flatness and call the mathematical flat surface a plane. To study properties of planes, we think of properties of flat surfaces. If we wish to draw a picture of a plane we draw something suggesting a flat surface.

Possibly the most fundamental property of the set of lines in space is what we shall call

Property I: *For any two distinct points in space, there is one and only one line containing the two points.*

We may think of this property as the "straight string" property or the "line of sight" property. For any two points (positions) in a room (with no obstructions), a string can be stretched between the two points (there is one line containing the two points). Any other string stretched between the two points would occupy the same place as the first string (there is only one line containing the two points). If A and B are points, we use the symbol $\overleftrightarrow{AB}$ to denote the line containing A and B.

We might note here that another important property follows from our Property I: i.e., can be proved on the basis of Property I.

Property I-A. *If two distinct lines intersect (have a non-empty intersection), then the intersection is exactly one point.*

Proof: Suppose the two distinct lines ℓ_1 and ℓ_2 are such that $\ell_1 \cap \ell_2$ contains the two distinct points P and Q. By Property I, only one line can contain both P and Q. Therefore ℓ_1 and ℓ_2 must be the same line; i.e., ℓ_1 and ℓ_2 must be different names for the same line. This contradicts the fact that ℓ_1 and ℓ_2 are distinct and therefore completes the proof.

We next state a property relating the set of all lines with the set of all planes.

Property II: *If a line contains two points of a plane, it lies in the plane.*

We could alternatively say that the line is a subset of the plane or is contained in the plane. This property practically describes what we mean by a surface being flat. We might say that a surface is flat if for each pair of points of it the line joining them lies in the surface.

Note that any plane must extend indefinitely far, for it contains lines which do.

Property II gives us a property of the set of planes. It tells us something about what planes are like (in terms of lines). It does not say what will determine a plane. To assert what is sufficient to determine a plane we have

Property III: *For any three distinct points not all on the same line, there is one and only one plane containing the three points.*

Note the similarity between Properties I and III. Property I says that if A and B are points and A is not the same as B, then there is a unique line containing A and B. Property III says that if A, B, and C are points and there is no line containing A, B, and C, then there is a unique plane containing A, B, and C.

Property III might be called the "three-legged stool" property. If you hold a three-legged stool up in a fixed place, a flat surface can be held against the three tips of the legs (there is one plane containing the three points). Furthermore, any flat surface held against the three tips must coincide with the first one (there is only one plane containing the three points).

There is an interesting property which follows from Properties I, II, and III; i.e., is implied by Properties I, II, and III.

Property III-A. If the intersection of two distinct planes contains two distinct points, then the intersection must be a line.

Proof: Let M_1 and M_2 be the distinct planes such that $M_1 \cap M_2$ contains the distinct points P and Q. By Property I, there is a unique line (call it ℓ) containing P and Q. By Property II, ℓ is a subset of M_1 and also is a subset of M_2. Therefore $M_1 \cap M_2$ contains the line ℓ. If $M_1 \cap M_2$ contained any point R not on ℓ, then P, Q and R would be three points not on the same line (ℓ doesn't contain all three and any line other than

ℓ cannot contain even P and Q). Then M_1 and M_2 would be distinct planes containing the three points P, Q and R and Property III says this cannot happen. Therefore, $M_1 \cap M_2$ not only contains ℓ but _is_ ℓ; the intersection is a line. Thus Property III-A is proved.

Another useful property follows from those we have stated. Its proof is left to the exercises.

Property III-B. _If ℓ is a line and P is a point not on ℓ, then there is one and only one plane that contains P and ℓ._

Exercises

1. Suppose P, Q, and R are three distinct points and are all in each of two different planes. What can be said about P, Q, and R?
2. Suppose points P, Q, and R are in only one plane. What can be said about the line containing P and Q?
3. (a) Suppose three points are not all on the same line. How many different lines contain at least two of them?
   (b) Suppose four points are not all in the same plane. How many different planes contain at least three of them?
   (c) In (b) how many different lines contain at least two of them?

4. (a) How many different lines may contain one point? Two distinct points?
   (b) How many different planes may contain one point? Two distinct points? Three distinct points not on the same line?
5. Prove Property III-B.

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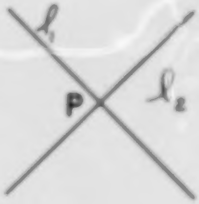
2. Intersections of Lines and Planes in Space. On the basis of Properties I, II, and III we are able to arrive at some conclusions concerning the nature of intersections of lines and planes in space. In fact, Properties I-A and III-A embody just such conclusions.

Case I: Intersection of Two Distinct Lines.

Let ℓ_1 and ℓ_2 denote two lines with $\ell_1 \neq \ell_2$

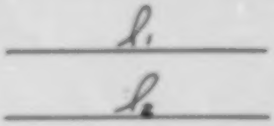
(a) Suppose $\ell_1 \cap \ell_2 \neq \emptyset$, i.e. $\ell_1 \cap \ell_2$ is not empty. Then by Property I-A, $\ell_1 \cap \ell_2$ is a set consisting of a single point. We shall show that $\ell_1 \cup \ell_2$ must be a subset of one plane. For let P be the point of intersection of ℓ_1 and ℓ_2. Let Q_1 be a point of ℓ_1 other than P and let Q_2 be a point of ℓ_2 other than P. Then P, Q_1, and Q_2 can not be on any one line and thus there is a unique plane



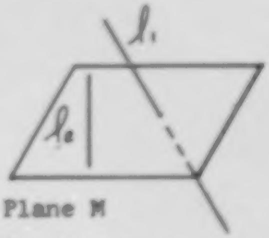
containing P, Q_1 and Q_2. But by Property II this plane must contain ℓ_1 (since it contains P and Q_1) and must contain ℓ_2 (since it contains P and Q_2).

(b) Suppose $\ell_1 \cap \ell_2 = \emptyset$, i.e., $\ell_1 \cap \ell_2$ is empty. Then one of two situations is true

i) ℓ_1 and ℓ_2 are subsets of the same plane. In this event, ℓ_1 and ℓ_2 are called parallel lines.



ii) ℓ_1 and ℓ_2 are not subsets of the same plane. Then ℓ_1 and ℓ_2 are called skew lines. Many pairs of skew lines are suggested by objects in a room. A "north-south" line on the ceiling and an "east-west" line on the floor are skew.



$M \supset \ell_2$. ℓ_1 pierces M. ℓ_1 and ℓ_2 are skew.

We might reorganize Case I as follows: If ℓ_1 and ℓ_2 are distinct lines, then either

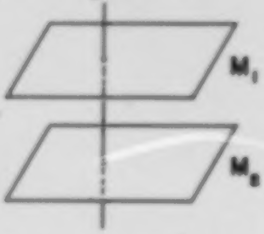
1) $\ell_1 \cup \ell_2$ is not a subset of any one plane. In this event, ℓ_1 and ℓ_2 are skew and $\ell_1 \cap \ell_2 = \emptyset$ or

2) $\ell_1 \cup \ell_2$ is a subset of some plane. If $\ell_1 \cap \ell_2 = \emptyset$, then ℓ_1 and ℓ_2 are parallel. If $\ell_1 \cap \ell_2 \neq \emptyset$, then $\ell_1 \cap \ell_2$ is one point.

Case II: Intersection of Two Distinct Planes.

Let M_1 and M_2 denote planes with $M_1 \neq M_2$.

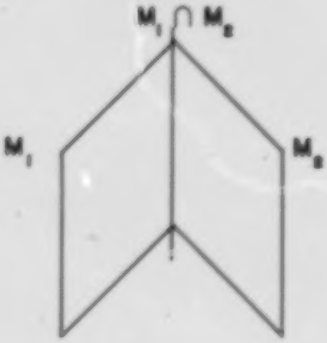
(a) Suppose $M_1 \cap M_2 = \emptyset$, i.e., M_1 and M_2 have no points in common. Then M_1 and M_2 are said to be parallel. Usually, planes of the floor and ceiling of a room are parallel.



(b) Suppose $M_1 \cap M_2 \neq \emptyset$, i.e., M_1 and M_2 do intersect. We need one more property of the set of planes in space to handle this case completely. This property like the others is intuitively rather clear.

Property IV. If two planes intersect, the intersection contains more than one point.

Therefore if $M_1 \cap M_2 \neq \emptyset$, $M_1 \cap M_2$ must contain more than one point. Thus by Property III-A, the intersection must actually be a straight line. Two positions of a door represent planes whose intersection would be the line through the hinges of the door.



We may summarize Case II,

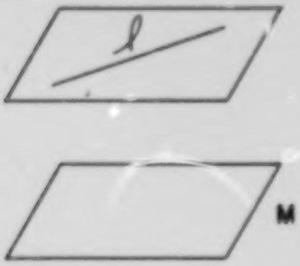
i) $M_1 \cap M_2$ is empty. Then M_1 and M_2 are parallel.

ii) $M_1 \cap M_2$ is not empty. Then $M_1 \cap M_2$ is a line.

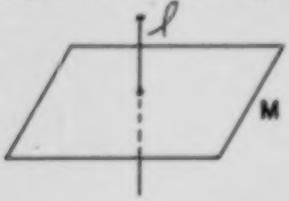
Case III: Intersection of a Line and a Plane.

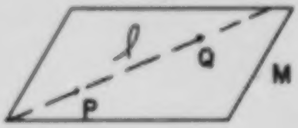
Let M be a plane and let ℓ be a line.

(a) Suppose $M \cap \ell = \emptyset$, i.e., M and ℓ do not intersect. We say that M and ℓ are parallel or that the line ℓ is parallel to the plane M. Any line in the plane of the ceiling is parallel to the plane of the floor.



(b) Suppose $M \cap \ell \neq 0$, i.e., M and ℓ do intersect. Then either $M \cap \ell$ consists of exactly one point or $M \cap \ell$ contains more than one point. In the latter case, by Property II, ℓ must lie in M or, in other words, $\ell \subset M$.





We may summarize Case III.

i) $M \cap \ell$ is empty.

ii) $M \cap \ell$ is one point.

iii) $M \supset \ell$.

Note that in all of these discussions we have not used the concept of distance or measure at all. We have been concerned with what are called "incidence relations", i.e. intersections of lines and planes.

In studying and understanding geometric considerations like those of this section, the teacher or student ought to think in terms of the geometry, that is, typical representations of the mathematical objects. He ought not to memorize facts as such, but rather he ought to get the geometric point-of-view through visualization. If he does, then he will know the "facts" without further effort because he will understand the intuition and spatial relationships behind this aspect of geometry.

Exercises

1. Describe two pairs of skew lines suggested by edges in your room.
2. On your paper, label three points A, B, and C so that $\overleftrightarrow{AB}$ is not $\overleftrightarrow{AC}$. Draw the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{AC}$?
3. Carefully fold a piece of paper in half. Does the fold suggest a line? Stand the folded paper up on a table (or desk) so that the fold does not touch the table. The paper

makes sort of a tent. Do the table top and the folded paper suggest three planes? Is any point in all three planes? What is the intersection of all three planes? Are any two of the planes parallel?

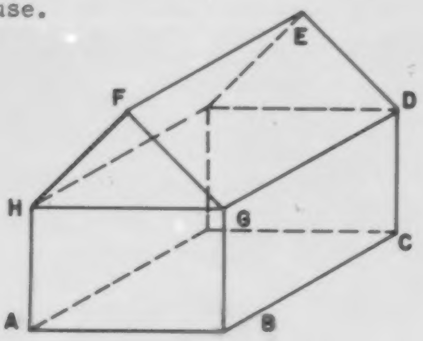
4. Stand the folded paper up on a table with one end of the fold touching the table. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?

5. Hold the folded paper so that just the fold is on the table top. Are three planes suggested? Is any point in all three planes? What is the intersection of the three planes?

6. In each of the situations of Exercises 3, 4, and 5 consider only the line of the fold and the plane of the table top. What are the intersections of this line and this plane? You should have three answers, one for each of 3, 4, and 5.

7. Consider three different lines in a plane. How many points would there be with each point on at least two of the lines? Draw four figures showing how 0, 1, 2, or 3 might have been your answer. Why could not your answer have been 4 points?

8. Consider this sketch of a house.

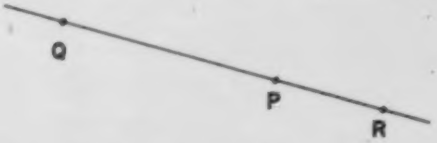


We have labeled eight points on the figure. Think of the lines and planes suggested by the figure. Name lines by a pair of points and planes by three points. Name:

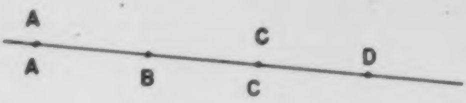
(a) A pair of parallel planes.
(b) A pair of planes whose intersection is a line.
(c) Three planes that intersect in a point.
(d) Three planes that intersect in a line.
(e) A line and a plane whose intersection is empty.
(f) A pair of parallel lines.
(g) A pair of skew lines.
(h) Three lines that intersect in a point.
(i) Four planes that have exactly one point in common.

3. Betweenness, Segments and Separations.

If P, Q, and R are 3 points of a line then it is intuitive that one must be between the other two. In the drawing P is between Q and R.



We shall assume betweenness properties of sets of points on a line without explicitly stating these properties. An example of such an assumption would be that as in the figure below, if C is between A and D and B is between A and C then B is between A and D and C is between B and D.



Euclid did not fully appreciate the significance of betweenness properties. It remained for geometers of the last hundred years to emphasize the fundamental nature of betweenness and its associated concept of order of points on a line.

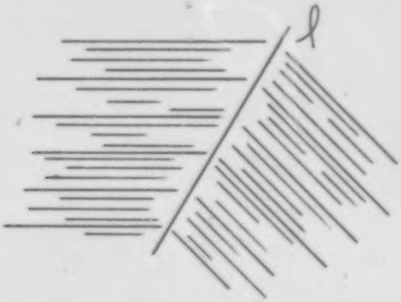
It is not the intention of this book to give a complete treatment of the foundations of geometry. Rather, here, we simply note the importance of the betweenness concept and tacitly assume what is geometrically evident about betweenness.

Let ℓ be a line and let P and Q be points of ℓ. Then the set of all points which are between P and Q together with the points P and Q is called the segment PQ. We use the notation $\overline{PQ}$ or $\overline{QP}$ to denote the segment. Note that $\overline{PQ} \subset \overleftrightarrow{PQ}$ (the segment $\overline{PQ}$ is a subset of the line $\overleftrightarrow{PQ}$). There will be many contexts in geometry when we will find it useful to talk about segments, and it is frequently necessary to distinguish between a line and a segment which is a part of it.

We next consider an important relationship which has three similar manifestations.

(a) If ℓ is a line and P is a point of ℓ then P separates ℓ into two half-lines. The set of points of the line other than P is the union of these two half-lines. These two half-lines do not intersect. We call P the boundary on ℓ of each of the two half-lines.

(b) If M is a plane and ℓ is a line in M, then ℓ separates M into two half-planes. The set of all points of M not on ℓ is the union of these two half-planes. These two half-planes do not intersect. We call ℓ the boundary in M of each of the two half-planes.

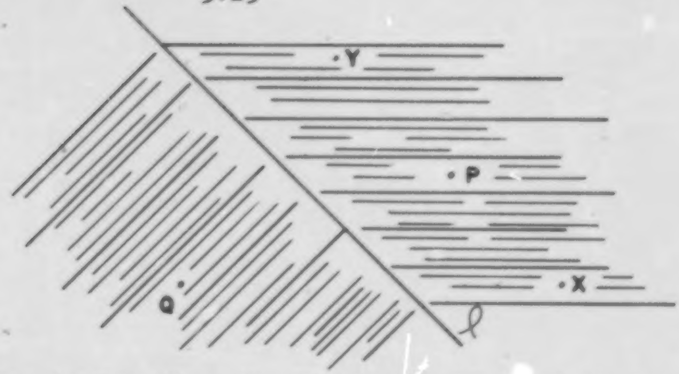


(c) If S is space (the set of points of space) and M is a plane (in S, of course) then M separates S into two half-spaces. The set of all points of S not in M is the union of these two half-spaces. These two half-spaces do not intersect.

We call M the boundary in S of each of the two half-spaces. Let us think of an example. The plane of the floor separates the set of points above the plane from the set of points below the plane.

One of the properties of these separations can be stated in terms of betweenness. We state it for the case of a line separating a plane (Case B).

5.15



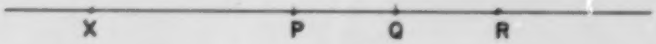
Let ℓ be a line in the plane M. Let P and Q be points in different half-planes determined by ℓ. Then there is a point of ℓ between P and Q.

Let X and Y be points in the same half-plane determined by ℓ. Then no point of ℓ is between X and Y. In other words, we have a criterion for determining whether two points of M not on ℓ are in the same half-plane bounded by ℓ. They are in the same half-plane if and only if no point of ℓ is between them. Analogous statements can be made in Case A of a point separating a line and in Case C of a plane separating space.

Sometimes in Case B of a line ℓ separating a plane M we call the half-planes bounded by ℓ the sides of ℓ (in M) and we denote the sides of ℓ by names of points in the sides. In the figure above we say the P-side of ℓ, the X-side of ℓ, the Y-side of ℓ or the Q-side of ℓ. Note that the first three of these are different names for the same set. The P-side of ℓ is the X-side of ℓ in our example. We also sometimes call the Q-side of ℓ the "non-P-side of ℓ".

Finally, we wish to introduce the term ray. If ℓ is a line and P is a point of ℓ then P separates ℓ into two half-lines (neither containing P). A set of points consisting of either of these half-lines together with P is called a ray of the line. The point P is called the endpoint of such a ray. We denote the ray as $\overrightarrow{PQ}$ where Q is some other point of the ray. In our notation $\overrightarrow{PQ} \neq \overrightarrow{QP}$.

Note that for the line in the figure:



(1) $\overrightarrow{XP} = \overrightarrow{XQ} = \overrightarrow{XR}$.

(2) $\overrightarrow{XP} \cap \overrightarrow{PX} = \overline{PX}$.

(3) $\overrightarrow{XP} \cap \overrightarrow{PQ} = \overrightarrow{PQ}$.

(4) $\overrightarrow{PX} \cap \overrightarrow{PQ}$ is the point P itself.

(5) $\overrightarrow{PX} \cup \overrightarrow{PQ} = \overleftrightarrow{PQ}$ (or $\overleftrightarrow{QR}$, etc.)

Exercises 5-3

1. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right. Name two segments
   (a) whose intersection is a segment.
   (b) whose intersection is a point.
   (c) whose intersection is empty.
   (d) whose union is not a segment.

2. Draw a line. Label three points of the line A, B, and C with B between A and C.

   (a) What is $\overline{AB} \cap \overline{BC}$?

   (b) What is $\overline{AC} \cap \overline{BC}$?

   (c) What is $\overline{AB} \cup \overline{BC}$?

   (d) What is $\overline{AB} \cup \overline{AC}$?

3. Draw a segment. Label its endpoints X and Y. Is there a pair of points of $\overline{XY}$ with Y between them? Is there a pair of points of $\overleftrightarrow{XY}$ with Y between them?

4. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.

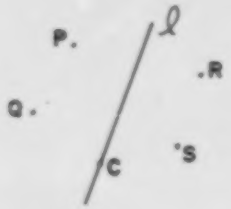
5. Draw two segments $\overline{PQ}$ and $\overline{RS}$ for which $\overline{PQ} \cap \overline{RS}$ is empty but $\overleftrightarrow{PQ}$ is $\overleftrightarrow{RS}$.

6. Let A and B be two points. Is it true that there is exactly one segment containing A and B? Draw a figure explaining this problem and your answer.

7. In some older geometry books the authors did not make any distinction between a line and a segment. They called each a "straight line". With "straight line" meaning either of these things, explain why we cannot say that "through any two points there is exactly one straight line."

8. Consider the figure at the right.

   (a) Is the R-side of ℓ the same as the S-side of ℓ?



(b) Is the R-side of ℓ the same as the Q-side?

(c) Are the intersections of ℓ and $\overline{PQ}$, ℓ and $\overline{RS}$ empty?

(d) Are the intersections of ℓ and $\overline{QS}$, ℓ and $\overline{PR}$ empty?

(e) Considering the sides of ℓ, are the previous two answers what you would expect?

9. Draw a line containing points A and B. What is $\overrightarrow{AB} \cap \overrightarrow{BA}$? What is the set of points not in $\overrightarrow{AB}$?

10. Draw a horizontal line. Label four points of it A, B, C, and D in that order from left to right.

Name two rays (using these points for their description):

(a) Whose union is the line.

(b) Whose union is not the line but contains A, B, C, and D.

(c) Whose union does not contain A.

(d) Whose intersection is a point.

(e) Whose intersection is empty.

11. Does a segment separate a plane? Does a line separate space?

12. Draw two horizontal lines k and ℓ on your paper. These lines are parallel. Label point P on ℓ. Is every point of ℓ on the P-side of k? Is this question the same as "Does the P-side of k contain ℓ"?

13. The idea of a plane separating space is related to the idea of the surface of a box separating the inside from the outside. If P is a point on the inside and Q a point on the outside of a box, does $\overline{PQ}$ intersect the surface?
14. Explain how the union of two half-planes might be a plane.
15. If A and B are points on the same side of the plane M (in space), must $\overleftrightarrow{AB} \cap M$ be empty? Can $\overleftrightarrow{AB} \cap M$ be empty?

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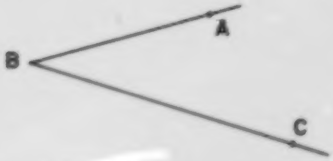
4. Angles and Parallel Lines.

Let A, B, and C be three points not all on the same straight line.

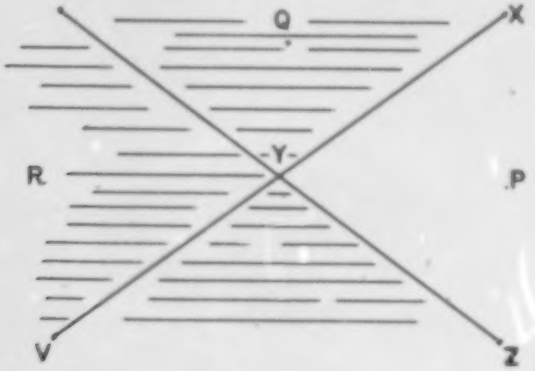


Then by Property III of Section 1, there is a unique plane which contains A, B, and C. By Property II of Section 1, the plane which contains A, B, and C also contains the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$ and, of course, all subsets of these lines.

The set $\overrightarrow{BA} \cup \overrightarrow{BC}$ (the union of the ray $\overrightarrow{BA}$ and the ray $\overrightarrow{BC}$) is called the angle ABC (or $\angle ABC$). B is called the vertex of the angle. The letter designating the vertex is always written as the middle of the three letters denoting the angle. We note that $\angle ABC = \angle CBA$ but $\angle ABC \neq \angle ACB$. By the definition above an angle is a set of points and is a subset of a plane. In Chapter 6 we shall deal with measures of angles but for the time being we are only concerned with an angle as a set of points.

We could have, equivalently, defined an angle as the union of two rays not on the same line and with a common endpoint. Note that this definition rules out "straight angles" and "zero-degree angles" as angles. Some people (and some mathematicians) may object to this restrictive definition but because of its simplicity, the useful purposes this definition serves, and the difficulties inherent in other possible definitions, we choose to use it. In Chapter 11 (on the circle), arcs and central angles of various degree measures are discussed.

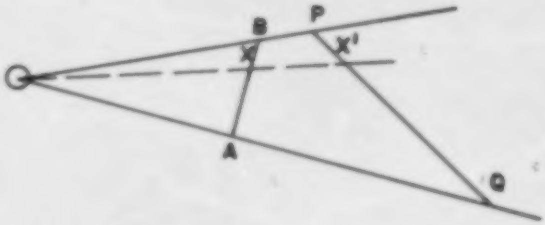
An angle (like a line) separates the plane of which it is a subset into two parts which are called the interior and the exterior of the angle. The angle is not in either part. The shaded portion below is the exterior, the unshaded portion the interior of ∠XYZ.



To be precise, we define the interior of the $\angle XYZ$ as the intersection of the two half-planes, the Z-side of $\overleftrightarrow{XY}$ and the X-side of $\overleftrightarrow{YZ}$. In the drawing the point P is in the interior of the angle for P is on the Z-side of $\overleftrightarrow{XY}$ and is on the X-side of $\overleftrightarrow{YZ}$. The exterior of the angle $\angle XYZ$ is defined to be the set of all points of the plane which are not on the angle or in its interior. The points Q, R, and V are all in the exterior of the angle.

Two angles are said to be vertical if their union is the union of two lines. Two angles are said to be supplementary (or supplement each other) if their union is the union of a line and a ray. (In other contexts, it will be convenient to say that two angles are supplementary if the sum of their degree measures is 180. They need not be "adjacent".)

Suppose $\overline{AB}$ and $\overline{PQ}$ are two segments as in the figure. We suppose $\overleftrightarrow{PB} \cap \overleftrightarrow{AQ}$ is the point O.



We wish to establish a one-to-one correspondence between the set of points of $\overline{AB}$ and the set of points of $\overline{PQ}$. For each point X of $\overline{AB}$, let X' be the point of $\overline{PQ}$ on the ray $\overrightarrow{OX}$. For each point of $\overline{AB}$ there is exactly one such ray and on each such ray containing a point of $\overline{AB}$ there is exactly one point of $\overline{PQ}$. Furthermore each point of $\overline{PQ}$ is on one such ray. Hence by use of these rays through O, we have a one-to-one correspondence between the set of points of $\overline{AB}$ and the set of points of $\overline{PQ}$.

We might also note that the consideration above also gives a one-to-one correspondence between (i) the set of points of the segment $\overline{AB}$ and (ii) the set of rays each of which has its end-point at O and lies in the set which is the union of $\angle BOA$ and its interior. We might describe the correspondence thusly:

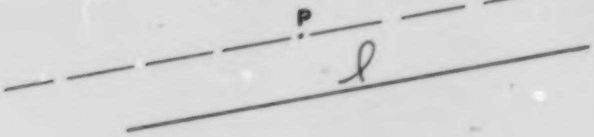
for x any point of $\overline{AB}$,

$$X \longleftrightarrow \overrightarrow{OX}.$$

Parallel Lines. It has already been observed in Section 2 that if two lines are in the same plane and do not intersect then they are said to be parallel. The concept of two lines being parallel does not involve measurement; it involves non-intersection of the pair of lines which are in the same plane. However, most criteria for determining whether two lines are parallel involve concepts of measurement: of equal distances or of congruent angles.

Historically, Euclid stated his famous parallel postulate which, rephrased, asserts

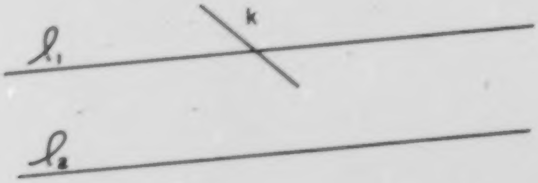
Property V: If ℓ is a line and P is a point not on ℓ, then in the plane containing P and ℓ there is one and only one line which contains P and does not intersect ℓ.



In Euclidean geometry, this property is regarded as intuitively clear.

We may deduce several other properties from Properties I - V.

Property V-A. *If* ℓ_1 *and* ℓ_2 *and k are three distinct lines in a plane* M, ℓ_1 *and* ℓ_2 *are parallel and* k *intersects* ℓ_1, *then* k *intersects* ℓ_2.



Proof: Let P be the point of intersection of ℓ_1 and k. Then by Property V, there is only one line in M which contains P and does not intersect ℓ_2. But ℓ_1 is such a line. Therefore k must intersect ℓ_2.

We might note that if ℓ_1 and ℓ_2 are parallel lines and k is a line in space which intersects ℓ_1, k need not intersect ℓ_2 for k and ℓ_2 might be skew lines.

Property V-B: *If* ℓ_1, ℓ_2 *and* k *are three distinct lines in a plane*, ℓ_1 *and* ℓ_2 *are parallel and* k *is parallel to* ℓ_1, *then* k *is parallel to* ℓ_2.

Proof: If k intersected ℓ_2 then by Property V-A, k would intersect ℓ_1 also and k is given as parallel to ℓ_1.

A property like V-B but without the restriction that the lines all be in a plane is also true. The argument is more complicated than that given. For instance, it is necessary to prove that k and ℓ_2 must be in the same plane.

There are some contexts in which we want to talk about segments or rays being parallel. Two segments or a segment and a ray or two rays are said to be parallel if the lines containing these segments or rays are parallel. A parallelogram, for instance, is a simple closed curve which is the union of four segments with each parallel to some other. Sometimes the symbol "||" is used to mean "parallel". For example, $\overline{AB} \parallel \overline{PQ}$ means that the segments $\overline{AB}$ and $\overline{PQ}$ are parallel to each other. The symbol ▱ is used to denote a parallelogram in the same sense that $\triangle$ is used to denote a triangle.

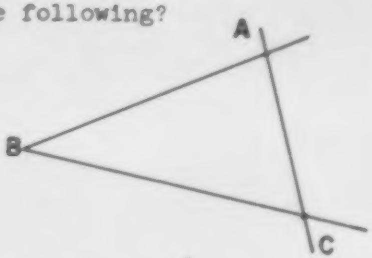
Exercises 5-4

1. Label three points X, Y, and Z not all on the same line.
   (a) Draw $\angle XYZ$ and $\angle XZY$. Are they different angles? Why?
   (b) Is $\angle YXZ$ different from both the angles you have drawn?
2. If possible, make sketches in which the intersection of two angles is
   (a) the empty set.
   (b) exactly two points.
   (c) a segment.
   (d) a ray.

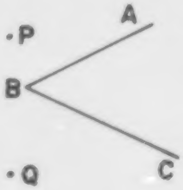
3. Draw two angles such that the interior of one contains the other.

4. (a) If two angles have a vertex and ray in common, must their interiors have a non-empty intersection?
   (b) If three angles have a vertex and ray in common must the interiors of some two of them have a non-empty intersection?

5. In the figure, what are the following?
   (a) $\angle ABC \cup \overline{AB}$.
   (b) $\angle ABC \cap \overline{AC}$.
   (c) $\overleftrightarrow{BA} \cap \overleftrightarrow{AC}$.
   (d) $\angle ABC \cup \overline{BC}$.

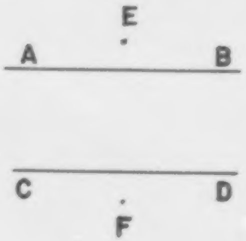


6. (a) Express the exterior of $\angle ABC$ in the figure as the union of two half-planes.



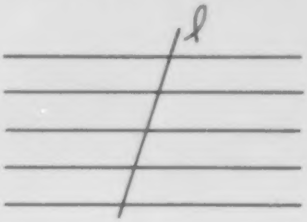
   (b) Draw a figure like that above and shade first one and then the other of the two half-planes whose union is the exterior of $\angle ABC$.

7. (a) Into how many sets does the union of two parallel lines separate the plane.
   (b) Describe the sets of (a) in terms of half-planes. You may think of the figure to the right.

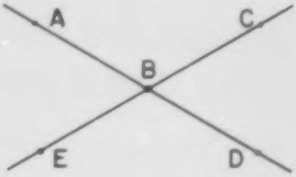


8. Using lines suggested by edges of a chalk box, give an example of two parallel lines and a line which intersects one but not the other.

9. Consider a set M of lines consisting of all lines in a plane parallel to (or the same as) a given line in the plane. For example, M might be the set of all horizontal lines on the plane of a chalkboard. Describe a one-to-one correspondence between M and the set of points of a line ℓ which intersects each line of M.



10. Using the figure on the right, list
    (a) all pairs of vertical angles,
    (b) all pairs of supplementary angles.



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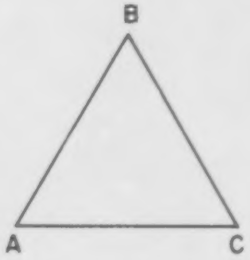
5. Special Subsets of Planes in Space.

Let A, B, and C be 3 points not all on the same (straight) line.

The triangle ABC (or $\triangle$ ABC) is the union of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. In notation,

$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}$.

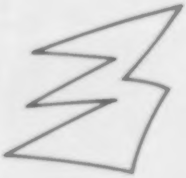


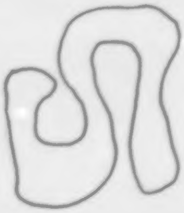
Thus a triangle is a set of points and is a subset of a plane. The points A, B, and C are called the vertices of the triangle and the angles $\angle ABC$, $\angle ACB$ and $\angle BAC$ are called the angles of the triangle ABC. Note that an angle of a triangle is not a subset of the triangle. An interior of an angle is not a subset of the angle nor is the boundary of a half-plane a subset of the half-plane. It is very common in mathematics as well as in ordinary language to use terminology like this. For example, we say "a radius and a center of a circle" but neither is a part of the circle. We speak of a triangle having an area but the area (which is a number of square units) is not a subset of the triangle but rather a number associated with the triangle. Thus our use of language is consistent with previous usage.

It is intuitively rather clear what we would mean by the interior of the ΔABC. The interior of Δ ABC can easily be defined as the intersection of the three half-planes: The A-side of $\overleftrightarrow{BC}$, the B-side of $\overleftrightarrow{AC}$ and the C-side of $\overleftrightarrow{AB}$. The interior is a set of points. The intersection of a triangle and its interior is empty. The exterior of Δ ABC is the set of all points of the plane containing A, B, and C which are not on the triangle or in its interior. We could also say that the exterior of the Δ ABC is the union of the non-C-side of $\overleftrightarrow{AB}$, the non A-side of $\overleftrightarrow{BC}$ and the non B-side of $\overleftrightarrow{AC}$.

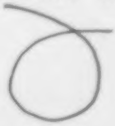
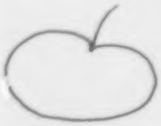
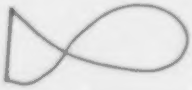
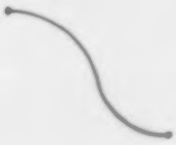
In geometry, there are many other figures, like triangles, which naturally arise. You are familiar with quadrilaterals, pentagons, rectangles, circles, etc. Note that the latter two of these involve concepts of measure. The rectangle involves the concept of a right angle (measurement of an angle) and a circle involves the concept of a length (the radius) and hence measurement of a segment. It is convenient to have one term which refers to all figures like those mentioned in this paragraph. We use the expression "simple closed curve". An accurate definition of "simple closed curve" involves concepts beyond those we choose to introduce here. But for our purposes we may think of a simple closed curve in a plane as a set of points which may be represented by a figure drawn in the plane without lifting the pencil, with the first and last points drawn coinciding but with no other points coinciding.

Examples of figures which represent simple closed curves are the following:





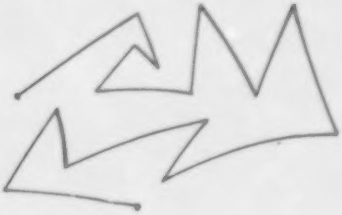
Examples of figures which do not represent simple closed curves are the following:

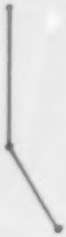





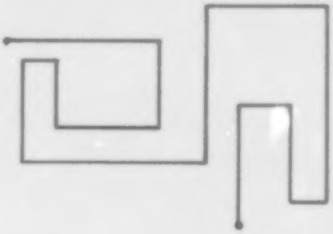
5.29

One of the important geometric theorems of the past century is the theorem that every simple closed curve in a plane separates the plane into two sets, an interior and an exterior. The simple closed curve is the boundary of each. We call the interior or the exterior (or a similar set) a region in the plane.

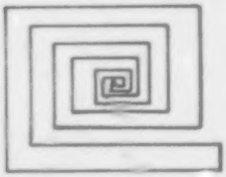
A polygonal path (or broken-line path) is a union of segments T_1, T_2, ---, T_n such that each has an endpoint in common with the following one and there are no other intersections. Examples of polygonal paths are:

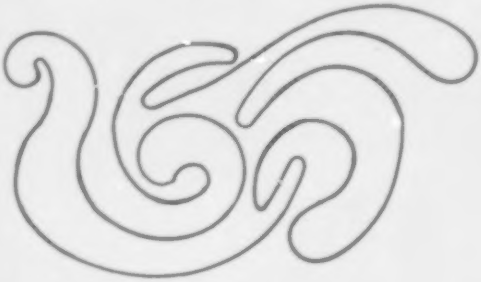






Note that in either figure below, it is not easy to tell whether a point is in the interior or the exterior or even if there is an interior or an exterior. One can observe the interior or exterior by shading or coloring near the curve without crossing the curve. For any simple closed curve J in the plane, the plane is the



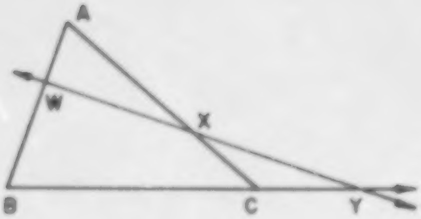


union of 3 sets no two of which intersect: the set J, the interior of J and the exterior of J. We can recognize whether two points P and Q not on J lie one in the interior and one in the exterior by the following criterion.

If every polygonal path (in the plane) from P to Q intersects J, then one of P and Q is in the interior and one is in the exterior. On the contrary, if some polygonal path from P to Q (in the plane) does not intersect J then P and Q are both in the interior or are both in the exterior.

Exercises 5-5

1. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.
    (a) Shade the C-side of $\overleftrightarrow{AB}$. Shade the A-side of $\overleftrightarrow{BC}$. What set is now doubly shaded?
    (b) Shade the B-side of $\overleftrightarrow{AC}$. What set is now triply shaded?
2. Draw a triangle ABC.
    (a) In the triangle, what is $\overline{AB} \cap \overline{AC}$?
    (b) Does the triangle contain any rays or half-lines?
    (c) In the drawing extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?
    (d) What is $\overleftrightarrow{AB} \cap \triangle ABC$?



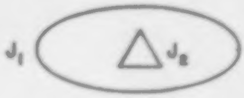
3. Refer to the figure on the right.
   (a) What is $\overline{YW} \cap \triangle ABC$?
   (b) Name the four triangles in the figure.
   (c) Which of the labeled points, if any, are in the interior of any of the triangles?
   (d) Which of the labeled points, if any, are in the exterior of any of the triangles?
   (e) Name a point on the same side of $\overleftrightarrow{WY}$ as C and one on the opposite side.

4. Draw a figure like that of Exercise 4.
   (a) Label a point P not in the interior of any of the triangles.
   (b) Label a point Q inside two of the triangles.
   (c) Label a point R in the interior of $\triangle ABC$ but not in the interior of any of the other triangles. (It can be done.)

5. If possible, make sketches in which the intersection of two triangles is:
   (a) the empty set.
   (b) exactly two points.
   (c) exactly four points.
   (d) exactly five points.

6. Draw a figure representing two simple closed curves whose intersection is exactly two points. How many simple closed curves are represented in your figure?

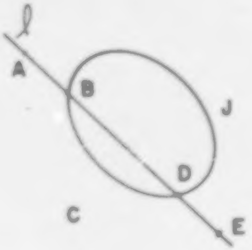
7. In the figure on the right, describe the region between the simple closed curves in terms of intersection, interior and exterior.



8. Draw two triangles whose intersection is a side of each. Is the union of the other sides of both triangles a simple closed curve? How many simple closed curves are represented in your figure?

9. Think of X and Y as bugs which can crawl anywhere in a plane. List three different simple sets of points in the plane any one of which will provide a boundary between X and Y.

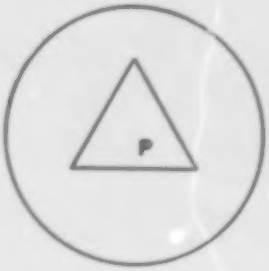
10. The line ℓ and the simple closed curve J are as shown in the figure.
    (a) What is $J \cap \ell$?
    (b) Draw a figure and shade the intersection of the interior of J and the C-side of ℓ.
    (c) Describe in terms of rays the set of points on ℓ not in the interior of J.



11. Draw two simple closed curves whose interiors intersect in three different regions.

12. Explain why the intersection of a simple closed curve and a line cannot contain exactly three points if the curve crosses the line when it intersects it.

13. (a) In the (plane) figure on the right describe a one-to-one correspondence between the set of rays with endpoint at P and the set of points of the triangle.



(b) Describe a one-to-one correspondence between the set of points of the triangle and the set of points of the other simple closed curve.

14. Draw two simple closed curves, one in the interior of the other such that, for no point P do the rays from P establish a one-to-one correspondence between the two curves.

Chapter 6

Measurement

1. Continuous Quantities and Length.

There are some numerical questions for which the correct answer, in the nature of things, must be a counting number or zero. How many children are in the 8th grade at your school? How many automobiles are registered in your state? In either case, a numerical answer which is not a whole number is ridiculous. A quantity for which a counting process as such is appropriate is called a discrete quantity.

There are some quantities--called continuous quantities--which require measuring and for which counting as such is inappropriate. How long is the house? How hot did it get yesterday? What is the area of the rug? Questions of this type have numerical answers which are obtained by measuring (or estimating measurements). Answers may be given in terms of whole numbers or they may involve rational numbers or fractions. Answers that are given are not absolutely precise as such. The accuracy of the number used is usually restricted by unevenness in the object measured, by the measuring instrument we use, and by our own intention in approximating an answer.

In Chapter 7, we shall investigate accuracy and precision of measurement in more detail. In this chapter we confine ourselves to the meaning of measurement.

Among quantities we measure are length, area, volume, angle size, temperature, speed, voltage and duration of time (to mention only a few). In this chapter and book we are primarily concerned with the measurement of geometric quantities like length, area, volume and angle size. Many of our observations are applicable to consideration of other quantities but we stress the geometric aspects.

In the previous chapter we observed that geometric space and its subsets like lines and planes were abstractions of physical objects in the world about us. In particular, lines were abstractions of straight edges (but without limits or endpoints). A segment (which is a subset of a line) is an abstraction of something like an edge of a box or a taut string stretched between two objects (points). If we want to measure the length of something in the physical world we have an analogous geometric problem of measuring the length of a segment. Thus in studying the process of measuring the lengths of physical objects we study the process of measuring segments in geometry and, even more important, we study the meaning of length in geometry (of distance between pairs of points). Our study of length in geometry, then, gives us insight and understanding of the measurement of length of any straight object in the world about us whether the object be a pin, a house, or the straight line path between two stars.

In what follows we try to develop fundamental relationships between the idea of congruence, the process of measurement, and the "coordinatization" of rays and other geometric sets. The approach is one of emphasizing concepts and developing understanding.

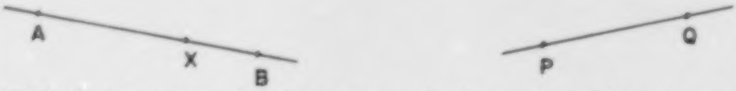
Length. Let $\overline{AB}$ and $\overline{PQ}$ be segments represented as below.

Our first consideration may well be to ask "Which is longer?" Later we might ask "Which is longer and by how much?" There is something intuitive about comparing two segments to see which is longer. But let us be more specific. "By what device can we compare them?"

In traditional Euclidean Geometry, there is a postulate to the effect that a geometric figure can be moved without changing its size or shape. This, when you really think about it, is a rather vague way of expressing an idea. What do we mean by "moving" a geometric figure? For a segment, we think of using a compass or a pair of dividers to "move" the segment. But even so, the motion--the process of moving a copy of a segment--isn't actually what we have in mind. A better way of describing what is meant might be to say that we can construct a copy of the figure (segment) near, on, or in relation to some other figure. Even this isn't really what we mean. From some points of view, the construction process is not important. What is important is that there *exists* a copy of the figure in any other place where we want it to be.

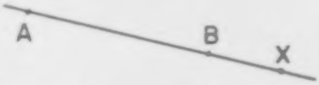
Now what do we mean by a *copy* of the segment $\overline{PQ}$? We mean a segment $\overline{P'Q'}$ which is "congruent" to $\overline{PQ}$, i.e., a figure of the same size and shape. In this treatment of geometry we choose to start with certain postulates about congruence which are assumed to be true. Our congruence postulates (properties) will concern segments and angles. This is more elementary than having congruence postulates concern all sorts of figures. We use the symbol "$\cong$" to mean "is congruent to".

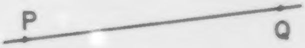
Property I: *Let $\overrightarrow{AB}$ be a ray and let $\overline{PQ}$ be a segment. Then there exists exactly one point X on the ray $\overrightarrow{AB}$ such that $\overline{AX} \cong \overline{PQ}$.*



Note that this property is a somewhat more explicit way of telling us that the segment $\overline{PQ}$ may be freely moved without changing its size or shape. Later, we shall see how, using this property, we can state a more general property about moving any geometric figure.

If, as in the case of our illustration, X is between A and B, then $\overline{AB}$ is longer than $\overline{AX}$ and hence longer than $\overline{PQ}$. If X were B (i.e., X and B were names for the same point) then $\overline{AB}$ and $\overline{PQ}$ would be equally long. If B were between A and X, then $\overline{AB}$ would be shorter than $\overline{AX}$ and as $\overline{AX} \cong \overline{PQ}$ we would say that $\overline{AB}$ would be shorter than $\overline{PQ}$.

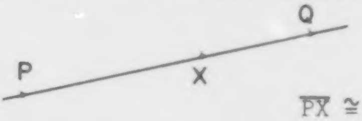




Thus Property I lets us compare any two segments as to length. In a full treatment of geometry, we should have to state other assumptions about comparing segments including, for instance, that if $\overline{PQ}$ is longer than $\overline{AB}$ and $\overline{AB}$ is longer than $\overline{RS}$ then $\overline{PQ}$ is longer than $\overline{RS}$. In this book, we shall tacitly assume such further properties without listing them. These properties concerning comparison of intervals are exactly what one would expect.

One other example of what we accept is that we can compare $\overline{AB}$ with $\overline{PQ}$ or compare $\overline{PQ}$ with $\overline{AB}$ giving the same result.

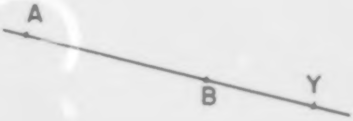
(1) We lay $\overline{AB}$ off on $\overrightarrow{PQ}$.



$\overline{PX} \cong \overline{AB}$

Therefore $\overline{PQ}$ is longer than $\overline{AB}$.

(2) We lay $\overline{PQ}$ off on $\overrightarrow{AB}$.

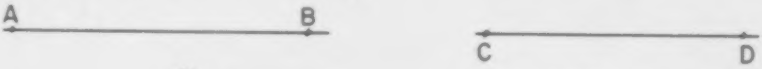


$\overline{AY} \cong \overline{PQ}$. Therefore $\overline{AB}$ is shorter than $\overline{PQ}$ and hence $\overline{PQ}$ is longer than $\overline{AB}$.

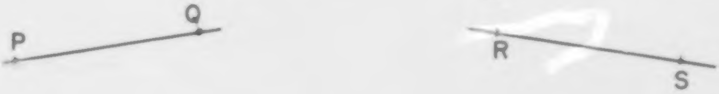
Exercises 6-1

1. List five "continuous quantities" not given in the text.
2. With a compass (or pair of dividers) compare the segments below with respect to length.

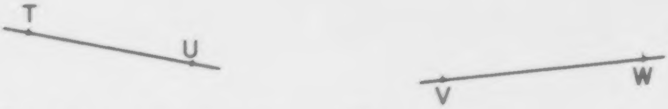
(a)



(b)



(c)



3. Is $\overline{AB}$ of the same length as $\overline{AB}$? Describe the process of comparing $\overline{AB}$ with itself.
4. Try to describe in your own words what is meant by saying that $\overline{AB}$ may be "freely moved". (Improve on the text if you can.)

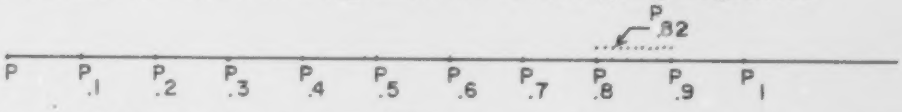
2. Properties of Length.

Let $\overline{PQ}$ be a segment. There exists a subdivision of $\overline{PQ}$ into two segments $\overline{PX}$ and $\overline{XQ}$ so that $\overline{PX} \cong \overline{XQ}$. This observation is tantamount to letting X be the midpoint of $\overline{PQ}$ (and asserts that such a midpoint X exists). Our intuition tells us also that there exists a subdivision of $\overline{PQ}$ into three non-overlapping congruent segments whose union is $\overline{PQ}$. (The segments are called "non-overlapping" if no two have any interior point of either in common.)

In fact our intuition further tells us that the following property should be true in geometry.

Property II. Let $\overline{PQ}$ be any segment. Then for any counting number k, there exist k non-overlapping segments whose union is $\overline{PQ}$ and such that all k segments are congruent to each other.

In dealing with the decimal representation of the real number system, we are particularly interested in the subdivision of a segment into 10 congruent subsegments, i.e., segments which are subsets of the original. For now we can begin to see how to associate real numbers with segments. The real numbers will represent lengths of segments. We think of the segment $\overline{PP_1}$ below as subdivided into 10 congruent non-overlapping segments.



We may think of the segment $\overline{PP_1}$ as having length 1. The segment $\overline{PP}_{.1}$ would have length 1/10 and $\overline{PP}_{.6}$, for instance, would have length 6/10.

Now each of the segments of length 1/10 indicated may be considered as similarly subdivided into 10 congruent subsegments. Thus, we have segments of length .01, .02, .03, ... and so on. For example, the segment $\overline{PP}_{.82}$ is of length .82.

Let us note a fundamental distinction in two different ways of saying something. If we say "Given a segment, we may subdivide it into 10 congruent subsegments", then we are forced to think of the process of subdividing a segment. We may feel that we could perform the process only a certain number of times. There might well be some last occasion at which we could perform it.

However, if we state the property in the form "Given a segment, there exists a subdivision of it into 10 non-overlapping congruent subsegments" then there is no process involving our own action or any time element. The subdivision exists whether or not there is any practicable way for us to do the subdividing.

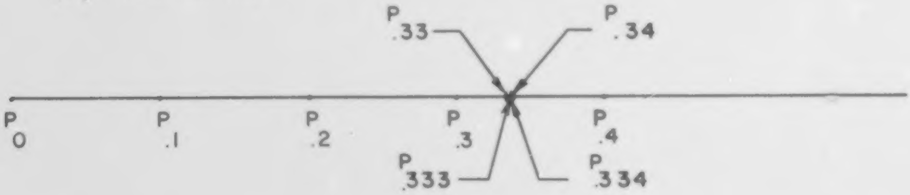
Thus we may speak of the number .3333 ---- (which is $\frac{1}{3}$) as being the length of the segment from P to that point

(a) which is in the segment from $P_{.3}$ to $P_{.4}$,

(b) which is also in the segment from $P_{.33}$ to $P_{.34}$,

(c) which is also in the segment from $P_{.333}$ to $P_{.334}$,

(d) and so on.



Clearly the next subdivision of any such segment into 10 congruent subsegments yields accuracy to the next decimal place. Thus the point $P_{.3333 \text{----}}$ should exist and the length of the segment from

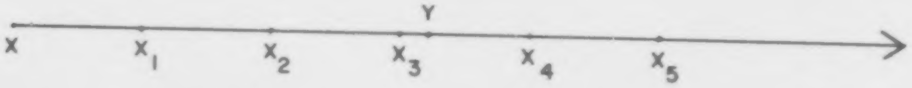
P to $P_{.3333\,---}$ is .3333 --- = $\frac{1}{3}$. This point happens to be the same point we get by subdividing $\overline{PP_1}$ into 3 congruent subsegments.

We have interpreted the positive real numbers less than 1 as lengths of segments laid off from P. It frequently is convenient to think instead of each point of the segment as corresponding to a real number--that number which represents the length of the segment from P to the point. By also using numbers greater than 1, we can similarly correspond the points of a ray to the positive (or zero) real numbers.

Another way of describing this point of view is to say that we are "coordinatizing" the ray. We are establishing a one-to-one correspondence between the set of points of the ray and the set of positive real numbers and zero. The point P corresponds to zero.

In order to assert the existence of the one-to-one correspondence which we are describing we need to note another basic property of geometry.

Let $\overrightarrow{XY}$ be a ray and let $\overline{AB}$ be a segment. Let X_1 be the point of $\overrightarrow{XY}$ for which $\overline{XX_1} \cong \overline{AB}$. Let X_2 be the point for which $\overline{X_1X_2} \cong \overline{AB}$ (by considering the ray with endpoint at X_1). Similarly let X_3 be the point for which $\overline{X_2X_3} \cong \overline{AB}$.



In this way points X_1, X_2, X_3, X_4, ... may be considered as existing.

Property III: *The ray $\overrightarrow{XY}$ is the union of the segments* $\overline{XX_1}$, $\overline{X_1X_2}$, $\overline{X_2X_3}$, $\overline{X_3X_4}$, ...

This property says that each point of the ray is in some segment $\overline{X_iX_{i+1}}$ or in other words that the successive reapplication of the segment $\overline{AB}$ to $\overrightarrow{XY}$ covers all of $\overrightarrow{XY}$.

Thus we see that the one-to-one correspondence between the set of points of the ray and the set of positive real numbers and zero can be set up as follows.

$$X \leftrightarrow 0$$
$$X_1 \leftrightarrow 1$$
$$X_2 \leftrightarrow 2$$
$$\vdots$$

The points of $\overline{XX_1}$ correspond to the real numbers from 0 to 1, the points of $\overline{X_1X_2}$ to the real numbers from 1 to 2, etc. The numbers are called the *coordinates* of the points.

The rather important observation we are now making is that for any positive (or zero) real number (i.e., a number which can be represented as a decimal expansion) there is a corresponding point of the ray $\overrightarrow{XY}$ and for any point of the ray $\overrightarrow{XY}$ there is a corresponding decimal expansion.

The positive (or zero) real numbers can be thought about in either of two equivalent ways:

(1) as denoting points of the ray $\overrightarrow{XY}$, or

(2) as denoting lengths of segments on $\overrightarrow{XY}$ with one end point of each at X. We let XY or m(XY) denote the length of $\overline{XY}$.

In (1) we are coordinatizing the ray. In (2) we are setting up the principles of measurement of length.

The coordinatization of the ray (or its analogue in (2)) involves three basic properties.

(a) Order is preserved. If P, Q, and R are 3 points of the ray and Q is between P and R, then the coordinate of Q is between the coordinates of P and R (as numbers).

(b) Distance is preserved. If $\overline{AB}$ and $\overline{PQ}$ are on the ray and $\overline{AB} \cong \overline{PQ}$, then the difference in the coordinates of A and B is equal to the difference in the coordinates of P and Q.

(c) Distance is additive. If B is between A and C, then AB + BC = AC.

The development we have here may be looked at in another way. It asserts the existence of a "ruler" in geometry. It says that a ray can be "coordinatized" and thus can be used as a ruler. Hence it says that a ruler exists and can be used.

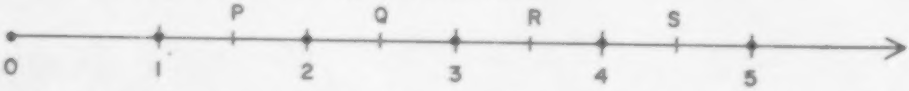
The length of a segment is thought of as a number--the unit in the geometric plane being understood. Note, then, that lengths can be added (because lengths are numbers and numbers can be added). We study measurement of lengths of geometric objects without reference to a unit. But in applying our knowledge of the principles of measurement to the every day world we are, in the nature of things, vitally concerned with the unit of measurement. The unit of measurement should always be specified in practical problems. We think of a length of a physical object as a certain number of units and the unit is specified. The number may be called the measure of the length of the physical object.

In light of these first two sections we now can observe that the statement "$\overline{AB} \cong \overline{PQ}$" is equivalent to the statement "AB = PQ". In other words if two segments are congruent, then their lengths are equal; and if two segments are of equal length, then they are congruent. We can use either type of language as convenient. Note, however, that the statement "$\overline{AB} = \overline{PQ}$" means something quite different from the other two. $\overline{AB} = \overline{PQ}$ means that $\overline{AB}$ is $\overline{PQ}$. As a consequence, A is P or Q and B is the other.

Exercises 6-2

1. Graphically describe the location of π to 3 decimal places. Bracket π between successive integers, tenths, hundredths and thousandths.
2. Do the same for $\sqrt{2}$.

3. (a) Using the figure below, give an example of statement (a) about order being preserved.



(b) Using the same figure, give an example of statement (b) about distance being preserved.

(c) Using the same figure, give an example of statement (c) about distance being additive.

4. If we subdivided segments into just 2 subsegments at each stage we would have a process suggesting the binary representation of the real number system. Explain and draw figures. (This problem is designed particularly for those who have some knowledge of the binary system. It could be used to develop such knowledge.)

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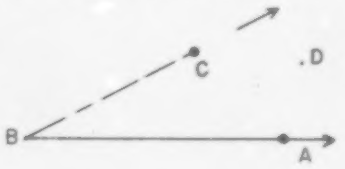
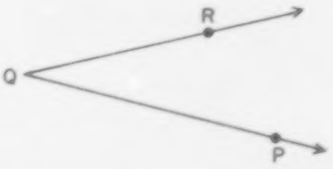
3. Angle Measure.

In the previous chapter we have defined an angle as a set of points, specifically as the union of two rays having the same endpoint and with the two rays not being on the same line. In the previous section, we introduced the concept of length or measure of a segment. In this section we similarly introduce the concept of measure of an angle.

In order to have a notion of size of angle (or angular measure) we first must have a notion of what we mean by saying that two angles are congruent (or have the same size) or that one angle is larger than the other. We could talk about moving angles around or constructing copies of them but, as before, we find it more convenient simply to assert the *existence* of certain angles.

Property I-A. *Let* $\angle PQR$ *be an angle. Let* $\overrightarrow{BA}$ *be a ray and let* D *be a point not on the line* $\overleftrightarrow{AB}$. *Then there exists exactly one angle,* $\angle ABC$, *such that* $\angle ABC \cong \angle PQR$ *and* C *and* D *are on the same side of the line* $\overleftrightarrow{AB}$ (*in the plane containing* $\overleftrightarrow{AB}$ *and* D).

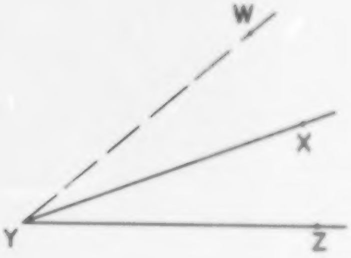


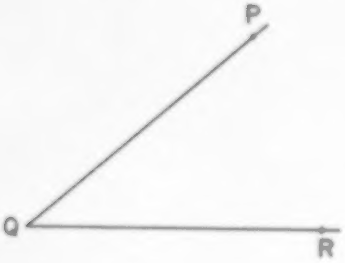


In old-fashioned terminology we can think of moving $\angle PQR$ so that ray $\overrightarrow{QP}$ falls exactly on ray $\overrightarrow{BA}$ and ray $\overrightarrow{QR}$ falls except for Q on the D-side of line $\overleftrightarrow{BA}$. Then the ray covered by $\overrightarrow{QR}$ would be $\overrightarrow{BC}$. Thus Property I-A is seen to be both intuitive and like properties of traditional geometry.

Properties I and I-A are quite similar. Each asserts the existence of exactly one figure of a given size starting from a given reference object (point or ray) and in a given "direction" from such reference object.

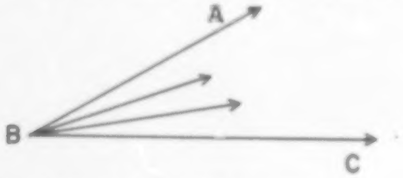
Property I-A tells us in effect how we can compare two angles to see which is larger.





Referring to the figure above, there is a copy of $\angle PQR$, such that $\overrightarrow{YZ}$ is a ray of the copy and the other ray $\overrightarrow{YW}$ lies (except for Y) on the X-side of $\overleftrightarrow{YZ}$. If X is in the interior of $\angle WYZ$ then $\angle PQR$ is larger than $\angle XYZ$. If X is on the ray $\overrightarrow{YW}$ then $\angle PQR \cong \angle XYZ$. And if X is in the exterior of $\angle WYZ$ then $\angle PQR$ is smaller than $\angle XYZ$. Thus Property I-A lets us compare two angles with respect to size.

Analagous to Property II we have the following Property (but it has to be stated a bit differently than was Property II).



Property II-A. *Consider* $\angle ABC$. *Let* k *be any counting number. Then there exist* k *congruent angles which subdivide the interior of* $\angle ABC$ *as follows*:

(1) *Each angle has* B *as a vertex*,

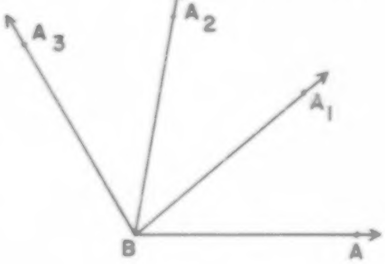
(2) *The interiors of the angles of the subdivision do not intersect*.

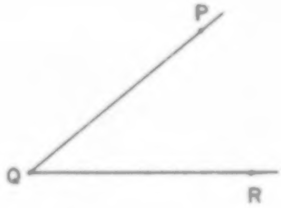
(3) *The union of the angles and their interiors is* $\angle ABC$ *together with its interior*.

Using this property, we can coordinatize the family of rays which have endpoint at B and lie on $\angle ABC$ or in the interior of $\angle ABC$. The process is like that of coordinatizing the set of points of a segment.

Finally we have a property which is like Property III in some respects but different in others.

Let $\overrightarrow{BA}$ be a ray and let $\angle PQR$ be an angle. Let $\overrightarrow{BA_1}$ be a ray such that $\angle A_1BA$ is congruent to $\angle PQR$.





Now consider ray $\overrightarrow{BA_1}$ and let $\overrightarrow{BA_2}$ be a ray with A_2 and A on opposite sides of $\overleftrightarrow{BA_1}$ such that $\angle A_2BA_1 \cong \angle A_1BA \cong \angle PQR$. Similarly there exists a ray $\overrightarrow{BA_3}$ such that $\angle A_3BA_2 \cong \angle PQR$ and A_3 and A_1 are on opposite sides of $\overleftrightarrow{BA_2}$. Thus there exist rays $\overrightarrow{BA_1}$, $\overrightarrow{BA_2}$, $\overrightarrow{BA_3}$, $\overrightarrow{BA_4}$, ... with similar properties.

Property III-A. *There is some number* n *such that* A_n *is not on the* A_1*-side of* $\overleftrightarrow{BA}$ *but all points* A_1, ..., A_{n-1} *are on the* A_1*-side of* $\overleftrightarrow{BA}$. *Furthermore there is some angle such that the point* A_2 *of this construction is on the line* $\overleftrightarrow{BA}$ *(but not on the ray* $\overrightarrow{BA}$*)*.

The first part of this property says that if you reapply any angle enough times, you will "get past" the other ray of the line you started with.

The second part of this property asserts that there is a right angle, i.e., an angle which is congruent to its supplement.

We say that two lines are perpendicular to each other if the union of two rays of these lines is a right angle. We use the symbol $\perp$ to mean "is perpendicular to".

As in the case of parallels, it is convenient to talk about lines, segments and rays being perpendicular to each other. For example, two rays or segments are perpendicular to each other if the lines containing them are.

From Property II-A, a right angle may be subdivided into 90 congruent angles whose interiors don't overlap. We speak of an angle of such a subdivision as an angle of one degree (or 1° in symbols). It follows from considerations like those for segments that any angle can be measured in terms of an angle of 1° and that the "degree measure" of an angle will be a positive number between 0 and 180.

Important Agreement. We agree to use the terms "degree measure of an angle" and "measure of an angle" synonymously. The measure of an angle, then, is a number between 0 and 180 and the degree symbol " $^\circ$ " need not be used. However, it is not "wrong" to use the degree symbol " $^\circ$ " and others may sometimes use it for emphasis or clarity. In notation, we write $m(\angle ABC)$ as the measure of $\angle ABC$.

An angle is said to be acute if its (degree) measure is less than 90 and to be obtuse if its (degree) measure is greater than 90.

It is not difficult to see that in applying Property II-A we could have used a subdivision of a right angle into any particular number of congruent angles. It is something of an historical accident that degree measure is used for expressing the size of an angle. We could just as well have used any angle as our basic unit (or reference) angle.

Another way of looking at the result of the coordinatization of the family of rays emanating from a given point and lying on one side of a line is to view the rays as in a protractor. We are saying that an (abstract) protractor exists as an instrument for measuring angles.

There are several important properties of geometry which may be considered as following from our assumptions here.

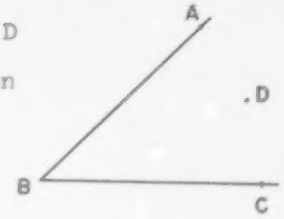
(1) The sum of the degree measures of an angle and its supplement is 180.

(2) If two angles have the same degree measure, they are congruent.

(3) Vertical angles are congruent (for they are supplements of the same angle and hence by (1) have the same degree measure and by (2) are, therefore, congruent.

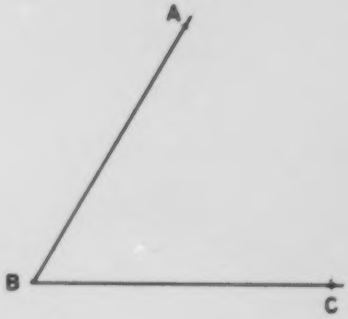
(4) If two angles are congruent, they have the same degree measure.

(5) Angle measure is additive. If D is in the interior of $\angle ABC$, then $m(\angle ABC) = m(\angle ABD) + m(\angle DBC)$.

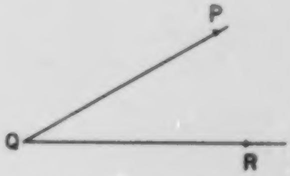


Exercises 6-3

1. Draw two angles and compare their "sizes" by the process of the text.
2. Draw an angle about like the one in the figure. Subdivide it into 6 congruent angles as in the text. You may use a protractor or do it approximately. The "size" of one of the angles of the subdivision bears what relation to the "size" of $\angle ABC$?



3. Draw an angle about like that in the figure. Draw a ray. Use the procedure of Property III-A and find the number "n" for this angle.



4. Try to restate Property II-A more simply.
5. Try to restate Property III-A more simply.
6. Illustrate by a specific numerical example what is meant by "Angle measure is additive."

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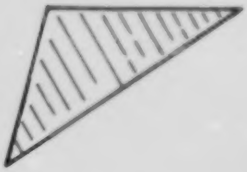
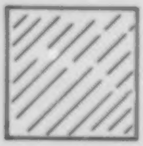
4. Area.

In the previous two sections we have developed the notions of linear measure (length) and angular measure. With respect to a standard segment (or angle) as a unit, any segment (or angle) can be measured. In this section, we consider another type of geometric object--a closed region--and try to measure it with respect to a standard closed region. Our early discussion is concerned with various types of closed regions. We lay down principles we shall want to use in later chapters. However, in this section we shall develop formulas only for rectangular regions.

Any simple closed curve in the plane is the boundary of its interior. The interior is sometimes called a region. We shall call the interior together with its boundary a closed region. Another way of saying this is that a closed region is the union of a simple closed curve and its interior.





The figures above represent closed regions. How can we compare two of them to see which is larger? The situation is not quite as simple as in the case of a segment or angle because the figures are not all directly comparable to each other. But we shall see in this section and in Chapters 10 and 11 how we can get around this difficulty.

We shall use the term "area" to describe our idea of the "size" of a closed region. The area of a closed region will be a number (or a number of standard units).

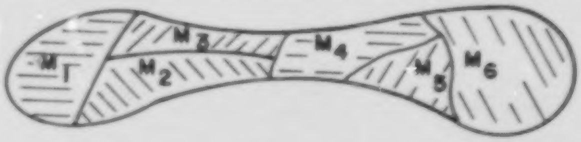
As in Sections 2 and 3 concerning congruences of segments and angles we make a number of fairly explicit assumptions about closed regions and area. All of the properties we shall state are what our intuition tells us to expect.

Property IV. Given a closed region, there exist closed regions congruent to it where appropriate, i.e., the closed region may be "freely moved" in the plane.

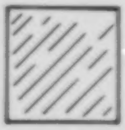
In Chapter 12,using the coordinate plane, we shall clarify the phrases "where appropriate" and "freely moved". For now we regard them merely as suggestive of the key idea.

Property V. If two closed regions are congruent to each other, then they have equal areas.

Property VI. Suppose a closed region is the union of non-overlapping closed regions. Then its area is the sum of the areas of the non-overlapping closed regions of which it is the union.



Now we come to the question of what we ought to use for a "standard" closed region. Several possibilities are represented in the figures below.

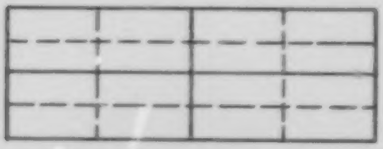


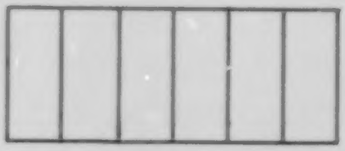
In the case of length and angular measure, all of the objects we were measuring looked comparable so this question did not arise.

A fundamental criterion of a "standard" closed region for area ought to be that the closed region can be expressed as the union of "small" congruent non-overlapping closed regions of the same general type.

If a closed region satisfies this criterion then we may break it up into small non-overlapping pieces and we would know how to break these pieces up into even smaller ones. A closed circular region is not suitable. We cannot easily break it up into smaller non-overlapping closed circular regions. Try it.

A rectangular region would satisfy the criterion.





There are many different ways in which we can express it as the union of smaller non-overlapping rectangular regions all congruent to each other. In fact, the rectangular region is delightfully suitable for our purposes and we shall use it. But to make things even easier, for our unit we shall use a special kind of rectangular region--a square region. We want the sides to be of equal length.

The square (or rectangular) region has another fortunate characteristic. It turns out that we can describe the area as the product of two lengths which are readily observable and measurable. Thus we can reduce many problems of computation of areas to problems of lengths.

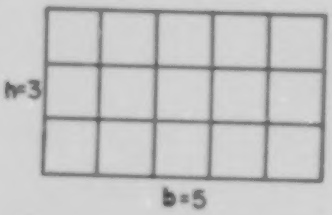
There is another assumption about area that we want to point out.

Property VII. With respect to a given rectangular region as a unit, then, for any other closed region, there is a unique area of this other region.

This property says that we must get the same answer no matter how we use our given square or rectangular region as a unit with respect to Properties IV - VI. We shall find it convenient to use one particular procedure. The answer we get is the same as that which we would get by different but legitimate procedures.

It is very common and convenient to talk about the area of rectangles, triangles, circles, and the like instead of talking about the areas of rectangular regions, for example. As long as we are aware that it is the region (and not the simple closed curve) which has the area there seems little confusion in using the traditional language. In what follows in this book, we shall use both types of terminology upon occasion, using the "region" language when there is need for emphasis on this concept.

The Area of a Rectangular Region. In dealing with rectangular regions we assume a unit length (or segment) to be given. The rectangular region has four sides. Opposite sides are of equal length. We may describe the rectangular region by giving two numbers which represent the lengths of adjacent sides. We use the notation (a by b) to denote a rectangular region with lengths of adjacent sides a and b. The properties of rectangles we use here and which we have not yet developed will be explained in Chapter 9.



Suppose we have a rectangular region which is b by h. In the figure b is the base and h is the height. We seek to express the area in terms of square units; i.e., in terms of a square region whose side is 1 unit.

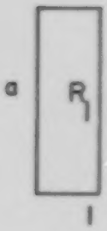
If b and h "come out evenly" in terms of whole numbers of our linear unit then the problem is easy. We can decompose the rectangular region into $b \cdot h$ non-overlapping square unit regions (all congruent to each other). In the figure $b = 5$ and $h = 3$, so the area is $b \cdot h$ or 15.

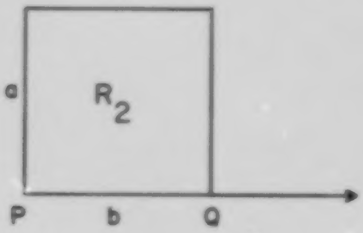
Our intuition is based on the "whole number" situation we have just considered. If, however, the base or the height is not a whole number of units the logical argument for the area as $b \cdot h$ is more complicated. The result, however, is still the same. We seek to justify the formula for any b and any h.

In the general case we are given two rectangular regions R_1 and R_2. We wish to express the area of R_2 in terms of the area of R_1. (Ultimately we are interested in considering R_1 as being a unit square region, but the argument is simpler without this assumption being made until later.)

We consider this problem in cases and in this way reduce a more complicated problem to two easy steps.

Case I. Suppose R_1 and R_2 have a side of each equal to a side of the other and further suppose the other side of R_1 is of length 1. R_1 is (a by 1) and R_2 is (a by b).

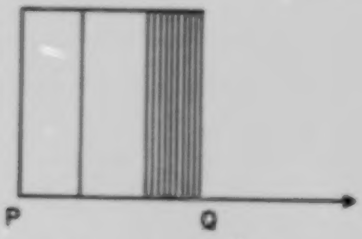




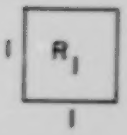
From Properties V and VI we know that if we regard R_1 as the union of 10 non-overlapping rectangular regions each being (a by $\frac{1}{10}$) then the area of any one of these is $\frac{1}{10}$ (Area R_1), (for the areas of these 10 regions must be equal and the sum of their areas must be (Area R_1).

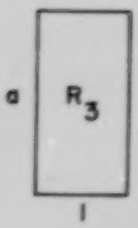
Similarly, the area of a rectangular region (a by $\frac{1}{100}$) is $\frac{1}{100}$ (Area R_1), and so forth.

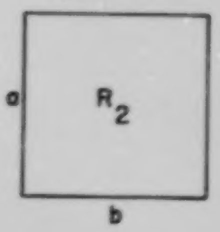
Now if P and Q are vertices of the base of R_2 we may regard $\overrightarrow{PQ}$ as coordinatized with unit length 1. Hence PQ = b. Consider the process of laying off non-overlapping copies of R_1 or R_2 starting from the left hand edge and then, having laid off all the copies of R_1 that are possible, we lay off copies of an (a by $\frac{1}{10}$) rectangular region in what is left of R_2, and then copies of an (a by $\frac{1}{100}$) rectangular region, and so on. This process is exactly equivalent to the process of finding the coordinate of Q, namely b, in terms of the unit length. In other words, (Area R_2) must be b · (Area R_1).



Case II. Suppose R_1 is (1 by 1) and R_2 is (a by b). We wish to express R_2 in terms of R_1.







We consider a rectangular region R_3 which is (a by 1).

Now from Case I, considering R_1 and R_3

$$(\text{Area } R_3) = a \cdot \text{Area } R_1,$$

and considering R_2 and R_3

$$\text{Area } R_2 = b(\text{Area } R_3).$$

But then

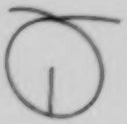
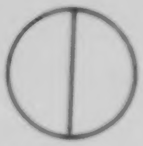
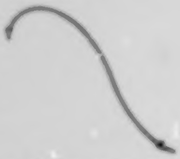
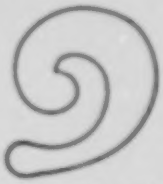
$$\begin{aligned}\text{Area } R_2 &= b(a \cdot \text{Area } R_1)\\ &= b \cdot a \cdot \text{Area } R_1.\end{aligned}$$

If we now agree to adopt a (1 by 1) square region as our unit then Area $R_2 = (b \cdot a)$ in terms of this unit. In our other symbolism, Area $R_2 = b \cdot h$.

Note that this gives us the usual formula for the area of a rectangular region in terms of the base and altitude (or height) of the region.

Exercises 6-4

1. Explain the distinction between an "area" and a "region".
2. Which of the figures below are the boundaries of regions which they determine?

3. If possible, express a triangular region as the union of four non-overlapping triangular regions all "congruent" to each other. (You will have to make a lot of implicit assumptions, some of which we will justify later.)

4. Suppose b and h are not whole numbers.
Explain in your own words why the area of a rectangular region (b by h) must be b · h.

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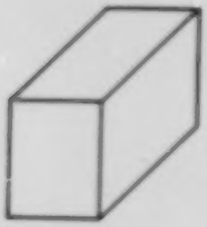
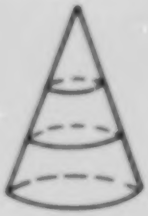
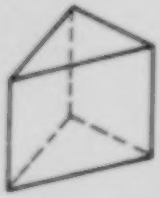
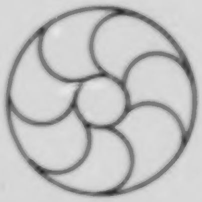
5. Volume.

In the preceding section we have observed some of the ideas underlying the concept of area. In this section we note that analogous considerations are applicable to the concept of volume.

A region in geometric space is the interior of a sphere (ball) or cube or such object. A closed region in space is the union of such a region and its boundary. The figures below can be considered to represent closed regions in space.

Associated with a closed region of such a type is a number (or a number of cubic units) called the volume of the region. In geometry the volume is a number whereas in practical problems a volume is expressed as a number of cubic units, there being some solid cube which is regarded as having unit volume.

Our first concern about size (volume) of two closed regions is to compare them to see which is larger. Comparisons of closed regions in space are even harder than comparisons of closed regions in the plane because of a greater diversity of types of figure.

However, as in the area case, it turns out that a rectangular figure is easiest and best to use for developing both the concept of volume and the computation of it. We use a rectangular parallelepiped (or box) for this purpose and ultimately use a cube as the simplest type of rectangular parallelepiped.

Technically, the terms cube and rectangular parallelepiped refer to the surfaces of solid objects in the same sense that square and rectangle refer to simple closed curves. But analogous to the language for area, it is common and convenient to refer to the volume of a cube (or parallelepiped or sphere or pyramid or such) instead of saying cubical region or spherical region, for example. Thus, when we say the volume of a cube we really mean the volume of the closed region in space bounded by the cube.

We have properties for volume analogous to those we have mentioned for area.

Property IV-A. *Given a closed region in space. There exist closed regions congruent to it where appropriate*; i.e., *the closed region may be "freely moved" in space*.

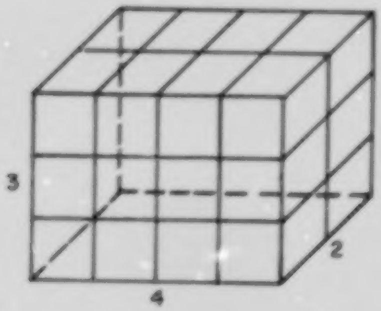
Property V: *If two closed regions in space are congruent to each other, then they have equal volumes*.

Property VI-A. *Suppose a closed region in space is the union of non-overlapping closed regions. Then its volume is the sum of the volumes of the non-overlapping closed regions of which it is the union.*

Property VII-A. *With respect to a closed rectangular space region as a unit, any other closed space region (of the type we are considering) has a unique volume.*

The Volume of a Rectangular Parallelepiped Region. The considerations here are like those of the preceding section with a cubical region of side 1 as our unit of volume.

A rectangular parallelepiped can be described by the lengths of three of its edges (no two of these three being parallel). We write (a by b by c). If each of a, b, and c is a whole number then by use of "building blocks" it is easy to see that the volume is $a \cdot b \cdot c$ or is $h \cdot B$ where we interpret a as the height h and B as the area of the base with b and c as the lengths of edges of the base.

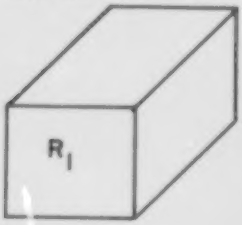


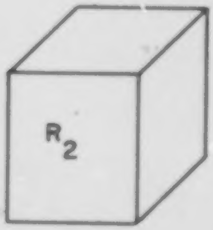
Clearly there are, in the figure, 8 unit blocks in each of three levels (tiers) and thus the volume is $3 \cdot 8$ or $3 \cdot 4 \cdot 2$.

The formula $V = a \cdot b \cdot c$ is what is usually used for computing the volume of a rectangular parallelepiped. The formula $V = h \cdot B$ is what is generalized to formulas for volumes of prisms, cylinders and the like.

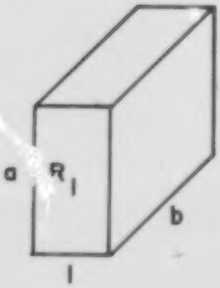
We now give a general proof of the formula $V = a \cdot b \cdot c$.

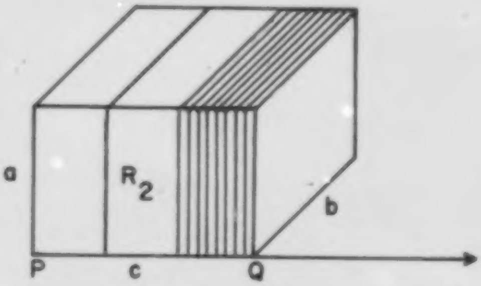
We are given two rectangular parallelepiped regions R_1 and R_2. We wish to express the volume of R_2 in terms of the volume of R_1.





Case I. Suppose R_1 and R_2 have two sides of each equal to two sides of the other and that R_1 has its other side equal to 1.





R_1 is (a by b by 1)

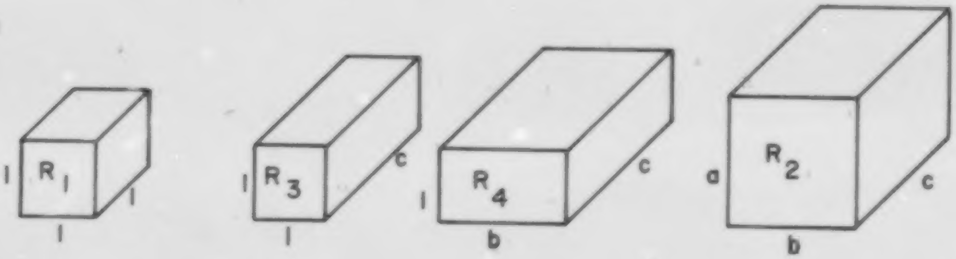
R_2 is (a by b by c)

From Properties V-A and VI-A it follows that a region $R_1(.1)$ of sides (a by b by .1) has volume equal to $\frac{1}{10}$ (Volume R_1), a region $R_1(.01)$ of sides (a by b by .01), has volume equal to $\frac{1}{100}$(Volume R_1) and so on. Hence in regarding R_2 as the union of copies of R_1 (starting from the left hand face) and then copies of $R_1(.1)$, and so on, we have that

$$(\text{Volume } R_2) = c \cdot (\text{Volume } R_1)$$

for the process is equivalent to that of laying off the unit segment in measuring $\overline{PQ}$.

Case II. Suppose R_1 is (1 by 1 by 1) whereas R_2 is (a by b by c).



We now use two intermediate regions, R_3 and R_4 with R_3 being (1 by 1 by c) and R_4 being (1 by b by c).

From Case I considering R_2 and R_4,

$$\text{Volume } R_2 = a \cdot (\text{Volume } R_4),$$

considering R_4 and R_3

$$\text{Volume } R_4 = b \cdot (\text{Volume } R_3),$$

and considering R_3 and R_1

$$\text{Volume } R_3 = c \cdot (\text{Volume } R_1)$$

Thus

$$\begin{aligned}\text{Volume } R_2 &= a \cdot (b \cdot (\text{Volume } R_3)) \\ &= a \cdot b \cdot (\text{Volume } R_3) \\ &= a \cdot b \cdot c \cdot (\text{Volume } R_1) \\ &= a \cdot b \cdot c\end{aligned}$$

if we agree to use R_1 as having unit volume.

Exercises 6-5

1. Explain the distinction between a "region in the plane" and a "region in space".
2. Explain why it would not be convenient to use a spherical closed region; i.e., the surface of a ball and its interior, as the unit of volume. (Refer to Section 4.)
3. Suppose a, b, and c are not whole numbers. Explain in your own words, why the volume of an (a by b by c) rectangular parallelepiped region must be $a \cdot b \cdot c$.

Chapter 7

Accuracy and Precision

1. The Significance of Numbers.

When we make a statement we try to convey some sort of information. We usually have three objectives in mind:

(1) to make a statement of some significance,

(2) to make a statement which is valid, and

(3) to make a statement which is not confusing; specifically, to make one which does not contain uselessly detailed or irrelevant information.

Unfortunately, it is frequently necessary to compromise between these various considerations. This is even true about statements involving numbers used to describe "counts" or "measurements" in practical situations. Furthermore, in making statements about counts or measurements we use many tacit understandings--some quite subtle--about what the numbers we are using mean. Many of these tacit understandings involve basic simple common sense. In this chapter we discuss common sense interpretations of the accuracy or precision of numbers as used both in counting and in measurement. The role of common sense in understanding the use and significance of numbers in counting and measurement cannot be overemphasized; it is impossible to lay down consistent, useful, hard-and-fast rules regarding the meaning of numbers and their significance.

Here we want to draw a clear-cut distinction between the principles of measurement in the abstract geometric plane as studied in the last chapter and the application of these principles to measurements in the everyday world. In Chapter 6 we have done "abstract" measurement to make it possible for us to understand basic concepts. In this chapter we restrict ourselves to statements and computations dealing with practical measurements (or counts). In Chapter 6, we could assert that the area of a geometric rectangle was equal to the product of the base times the height (Area = b · h). The numbers concerned were precise. In this chapter we can deal only with approximations and to emphasize this we shall use the symbol "$\approx$" to mean "is approximately equal to."

Most statements involving either counting numbers or measurements are, in the nature of things, not intended to be "precise" or "accurate". In many instances, they cannot be, if they are also going to be valid. While there may abstractly exist a precise count of a set of objects there may be no practicable way for humans to know what such count is. Consider questions involving

(a) the human population of the world (at this instant),

(b) the number of dollar bills in circulation, or

(c) the number of grains of sand on Waikiki beach.

Clearly we cannot give completely precise and valid answers to such questions. Furthermore, attempts to give completely precise answers would not only be incorrect but would also cause confusion and probably would lead to unnecessary and irrelevant arguments.

In the case of measurements we have an extra complicating factor. Practically, there is no "exact" measurement. Consider, for example, the length of a table, the area of a rug, the distance to the moon. The "objects" to be "measured" are uneven and must be. Even the standard "meter" in the Bureau of Standards is accurate only to a few decimal places. So we recognize that any numerical measurement givep must be in the nature of things, an "approximation".

In spite of such limitations of applications of our number system to problems of both counting and measurement, we still are led to understand a great deal about the physical world by our study of the "abstract" number system and "abstract" geometry and their uses in everyday life.

With considerations like the foregoing in mind we can better understand our use of numbers in both counting and measurement. We turn our attention specifically to measurements with the observation that our remarks restricted to whole numbers apply also to "counts".

7.4

Usually when we use numbers in measurements, we use them in one of two senses,

(a) at least this much, or

(b) closer to this number than to any other comparable one.

Examples where measurements may well be used with the tacit understanding of "at least this much" are

(i) 1 pound of hamburger,

(ii) a 15' pole vault,

(iii) a 6' man (in some senses) and

(iv) a 100° temperature (it was a hundred today).

Examples where measurements may well be used with the tacit understanding of "closer to this number than to any other comparable one" are

(i) a 6' table,

(ii) a 5'10" man,

(iii) a 15' room,

(iv) a 98° temperature (it is 98° outside now).

Depending on the contexts in which particular measurements are used, there may be differences of opinion as to "the" proper sense in which the number is meant.

In many instances, where numbers are used in the "at least this much" sense they are used as isolated numbers and computations are not made with them. If computations are going to be made--to find averages or complete areas, for example--the measurements are usually intended in the "closer to this number than to any other comparable one" sense. It is convenient for our purposes to agree on a convention concerning our use of numbers. With the full knowledge that the convention we adopt is not universally applicable, we agree to use the "closer to this number than to any other comparable one" meaning.

Greatest Possible Error. The greatest possible error in a measurement refers to the largest amount by which the given measurement differs from the "true" measurement of the object. In this discussion we assume proper use and reading of instruments. The "error" comes from the way we choose to (must) express our answer numerically. If we say that an object is 8' long, we mean usually that it is closer to 8' than to 7' or 9'. In other words, we mean that the "true" length is between 7.5' and 8.5'. In this case the "greatest possible error" is 0.5' (or 1/2 foot). The difference between the asserted length 8' and the "true" length is less than the greatest possible error (0.5' in this case).

Agreement. Unless the contrary is specified, the greatest possible error of a measurement given in decimal form is understood to be 1/2 of the place value of the last digit which is used in the numeral for a purpose other than simply locating the decimal point.

Let us consider some examples.

Numeral	Place value of last digit used for a purpose other than locating the decimal point	Greatest possible error
48.6	.1	.05
9800	100	50
.054	.001	.0005
830.00	.01	.005

Most readers probably have little question about the first and third examples. In the second, the two zeroes are considered used simply to locate the decimal point and hence neither is the "last digit to be considered". In the fourth example, the second and third zeroes are not used simply to locate the decimal point. They could be omitted. Hence they are considered used to indicate precision and the agreement gives .005 as the greatest possible error.

For numbers given in fractional form, the greatest possible error is understood to be 1/2 of 1/n where n is the denominator of the fraction. Thus a length of 6 7/8 inches is understood to have a greatest possible error of 1/16.

Exercises 7-1

1. Give three examples (of your own) of "counts" which cannot be precisely known.
2. Give three examples (of your own) of measurements used in the "at least this much" sense.

3. Give two examples of measurements used in a sense other than either of the senses (a) and (b) of the text. (One example might be "the 4 minute mile")
4. Discuss the following answers to the question "What was the population of New York City in 1950?" with respect to considerations (1), (2) and (3) at the beginning of this chapter.
   (a) 7,891,957 (the census figure)
   (b) 7,900,000
   (c) 8,000,000
   (d) 10,000,000
   (e) greater than 1,000,000 and less than 100,000,000

   Give contexts in which (a), (b), (c), and (d) would be reasonable answers.
5. Find the greatest possible error of each of the following measurements:
   (a) 93,000,000 miles
   (b) 820.1'
   (c) 16 1/4 inches
   (d) 3.460 miles
   (e) 71 yards
6. Explain why you should not make a statement like "This rug is 28.462 inches wide."

7. Explain how our convention on the greatest possible error of a measurement helps statements about numbers achieve some of the objectives listed at the beginning of this chapter.
8. Explain how 125,000 might be considered as having a greater possible error than 120,000. As a population of a city it probably would be so considered. Hint: 25 is 1/4 of 100.

2. Precision, Tolerance, Significant Digits, and Relative Error.

In the previous section, we have explained what we mean by the greatest possible error of a measurement. The precision of a measurement in decimal form is the place value of the digit we used in getting the greatest possible error. In other words, the precision is simply twice the greatest possible error. This technical meaning of the word precision agrees in principle with the everyday usage of the word. We might speak of a measurement which is precise (or accurate) to the nearest tenth of an inch, for example. (Later we shall give a technical meaning of the word "accurate".) We speak of a measurement of 8.24" as more precise than one of 63.9". If we were to ask, "How precisely do you want this measured?" we might expect an answer like "To the nearest tenth of an inch" or "to the nearest 1/4 of an inch".

There are many instances in which our agreement of the previous section on the greatest possible error is not suitable or convenient to describe the actual greatest possible error of a particular measurement. In such instances we may indicate the greatest possible error by stating it explicitly. Thus we might write 84.3" ±.02". We read the symbol "±" as "plus or minus" and we are saying that the "true" measurement is in between 84.3" - .02" and 84.3" + .02", therefore, between 84.28" and 84.32". The .02" is sometimes called the tolerance of the measurement. When the tolerance is important (as in machine shop work) it is very common to give it explicitly (even when it agrees with the convention we have established). One might write 3/8" ± .001". This indicates a measurement of 3/8 of an inch with an error of not more than a thousandth of an inch. We can conveniently combine fractions and decimals in this way and it is commonly done.

Another instance where our agreement on the greatest possible error does not always adequately deal with a situation is where several terminal zeroes are used in a numeral representing a whole number. Consider 180,000. Our agreement asserts that the greatest possible error is 5,000 (half of 10,000). But we would write the numeral exactly the same way if the greatest possible error were 500, 50, 5, or .5. We are saying that we can't really tell if some or all of the zeroes are intended to do more than just locate the decimal point. Sometimes the context of a statement tells us what is intended. We may use a bar over or under the right-most zero which is intended to be precise. Thus $180,\bar{0}00$ or $180,\underline{0}00$ has a greatest possible error of 50.

7.10

The considerations of the preceding paragraph suggest another concept that naturally comes up concerning numbers used in measurements. This concept is that of "a significant digit" in the decimal form of the number. A digit in a decimal numeral is spoken of as being a "significant digit" if it serves a purpose other than simply to locate (or emphasize) the decimal point. Some examples will clarify this:

Numeral	Significant digits (in order)
48030	4, 8, 0, 3
61.20	6, 1, 2, 0
841	8, 4, 1
0.00429	4, 2, 9
6.0031	6, 0, 0, 3, 1

In 48030, the "0" between the "8" and the "3" is significant; the other "0" is not, it simply locates the decimal point (understood). In the numeral 61.20, the "0" is significant because it is not necessary to have it to locate the decimal point. In 0.00429, all the zeroes are used simply to locate or emphasize the decimal point with the understanding that the leftmost zero may or may not be written and if written is simply for clarity in locating the decimal point and reading the number. It makes the decimal point stand out.

We are sometimes asked to count the number of significant digits in a numeral. We can be given instructions to "round off" numbers (i.e., numerals) in one of two ways; for instance,

(1) Round off a numeral to the nearest tenth, or

(2) Round off a numeral to three significant digits.

Consider 58.108. With respect to either of the instructions above the "rounded off answer" is "58.1". Rounding off the same numeral to four significant digits would yield "58.11". In the rounding off process we start from the right and move left. There may be ambiguity if the right-most non-zero significant digit is a five. Then we are at liberty to round off either to the lower or higher figure in the digit to the left of such five. We always ought to use all the information available in rounding off. For example, consider 437.496. Rounding off to 4 significant digits yields 437.5. Rounding off to 3 significant digits yields 437 for the value of .496 is less than ".5".

Relative Error. The concept of relative error is the concept of the relationship (specifically, the ratio) of the greatest possible error (sometimes called the absolute error) to the size of the number itself. Specifically, relative error is $\frac{\text{greatest possible error}}{\text{measured value}}$. The relative error is sometimes technically called the accuracy of the measurement. The more accurate the measurement the smaller the relative error. Let us consider two examples.

93,000,000 miles

$$\text{relative error} = \frac{500,000}{93,000,000} \approx .005$$

.03" ± .001"

$$\text{relative error} = \frac{.001}{.03} = \frac{1}{30} \approx .03$$

We can see that while the measurement on the right is far more "precise" (.001" to 500,000 miles) it is about 6 times less accurate (.03 to .005) than the other measurement.

The distinction between "greatest possible error" and "relative error" is an important one. The one we want to use depends on the context.

Exercises 7-2

1. Assume our agreement on greatest possible error. Explain the statement, "The more significant digits there are in a numeral the less the relative error." Use examples in your explanation if you wish.
2. State which of the following two measurements is more precise; is more accurate.
   (a) 68.3" and 12.34'
   (b) 82.01° and 0.014°
   (c) 16,000,000 light years and 1760 yards
   (d) 18 ± .3 and .8 ± .02.

3. How many significant digits are there in each of the following numerals?

   (a) 14.082

   (b) 9.600

   (c) 0.0316

   (d) 19,414,500

   (e) $16,\bar{0}00$

   (f) 0.00024

4. Round off to 3 significant digits

   (a) 4.86496

   (b) 13.021

   (c) 77,455,000

   (d) .0152897

5. What would be meant by the per cent of error in a measurement? How would it be related to the relative error?

6. Explain a situation where you would be interested in the relative error of a measurement.

7. Explain a situation where you would be interested in the greatest possible error of a measurement.

3. Precision and Accuracy in Computations Involving Addition.

We may frequently use measurements in various computations. Each number we use has a certain precision and a certain accuracy. We ask how precise or accurate the sum (or the product) of such numbers will be. The situation gets very complicated very rapidly. The best we can do here is to give some examples and suggest some reasonable "rules of thumb". Some understanding both of the nature of the problem and the limitations of our "rules" is necessary.

Suppose we want to add two numbers like 18.6 and 23.9. The greatest possible error is .05 in both cases. Below we have made some computations revealing the greatest possible error of the sum.

Least values		Greatest values
18.55	18.6	18.65
23.85	23.9	23.95
42.40	42.5	42.60

Thus the sum 42.5 really has a greatest possible error of 0.1; i.e., we know only that the "true" value is somewhere between 42.4 and 42.6. We could have written our computations as follows:

$$\begin{array}{lcl} 18.6 & \pm & .05 \\ 23.9 & \pm & .05 \\ \hline 42.5 & \pm & .10 \end{array}$$

In effect, we add the greatest possible errors of the addends to find the greatest possible error of the sum.

If we had three numbers 18.6, 23.9 and 41.2 to add together, then the greatest possible error of the sum would be 0.15. The more numbers we add together the less precise the answer can be asserted to be. However, it is impractical and inconvenient to state explicitly the greatest possible error of the sum. So as in the first illustration above, we would write our answer as 42.5 with the standard agreement that the "greatest possible error" is .05 but with the clear understanding that we cannot be certain about this much precision. In a sense we are "caught"; we have to compromise between technical validity of our statements and giving too many details.

If we have several measurements to add together, then the "law of averages" makes it unlikely that we will get the largest possible inaccuracy in each number in the same direction. In fact, we expect the "deviations" of the "true" measurements from the measurements we use to compensate for each other in part. Thus our use of the answer we get by ordinary straightforward calculation is really the best we can do and is likely to be fairly close to the "true" value.

Suppose we want to add 86 to 18.48. Here it simply does not make sense to write the answer as 104.48 for in so doing we are implying precision to the nearest .01 whereas the 86 presumably was precise only to the nearest unit. Thus we ought to write our answer as 104 or possibly as 104.5 with the .5 interpreted more as 1/2 than as 5/10. The "true" value is quite likely to be somewhere between 104 and 105 and thus 104.5 seems like a reasonable answer.

In bank statements and other financial accounts, a figure like $86 frequently means $86.00 and thus it is reasonable to add to the last cent if desired.

The question of accuracy in addition of measurements is even more complicated than that of precision. The sum is customarily more accurate than one of the addends and less accurate than the other. Consider the illustration:

$$\begin{array}{r} 104 \\ +\underline{\ 25} \\ 129 \end{array} \qquad \frac{.5}{104} \approx .005$$

$$\frac{.5}{25} \approx .02$$

$$\frac{1.0}{129} \approx .01$$

This indicates that the accuracy of the sum is about .01 which is between the computed accuracies of 25 and 104.

In subtraction problems, the accuracy of the difference may be far less than the accuracy of the other numbers used. (In other words, the relative error of the difference may be far greater than the other relative errors.) Consider the example below:

$$\begin{array}{r} 62 \\ -\underline{\ 58} \\ 4 \end{array} \qquad \frac{.5}{62} \approx .01$$

$$\frac{.5}{58} \approx .01$$

$$\frac{1.0}{4} \approx .25$$

Here the relative error is large because the difference (under subtraction) is a small number.

Exercises 7-3

1. Find the greatest possible error of the sum of
   (a) 180, 160, 140, and 80.
   (b) 16.8 ±.001 and 12.5 ± .002.

2. Work out the actual greatest possible error in (86 + 18.48) as in the text. (Hint: write 86 as 86 ± .5 and 18.48 as 18.48 ± .005.)
3. Give an illustration explaining the greatest possible error in a subtraction problem. Are the considerations like those for addition?
4. Find the relative error of the sum of
   (a) .023, .060, and .055.
   (b) .28 ± .01 and .42 ± .02.
5. Find the relative error of the difference of
   (a) .34 and .24.
   (b) 160 ± .1 and 100 ± .1.

4. Precision and Accuracy in Computations Involving Multiplication.

The situations relative to the greatest possible error and the relative error in multiplication (and division) are even less satisfactory than those in addition and subtraction. It might be observed that the subject of "error theory" is one which is being studied by mathematicians at the present time. The wide-scale use of computing machines makes "error theory" of great importance today.

If we multiply two measurements together, what can we say about the precision of the product? For instance, how many square feet are there in a room which is 16 ft. by 18 ft.? Most of us would say "288 sq. ft." but how precise is our answer? We assume (by our agreement) that 16 and 18 are precise to the nearest unit.

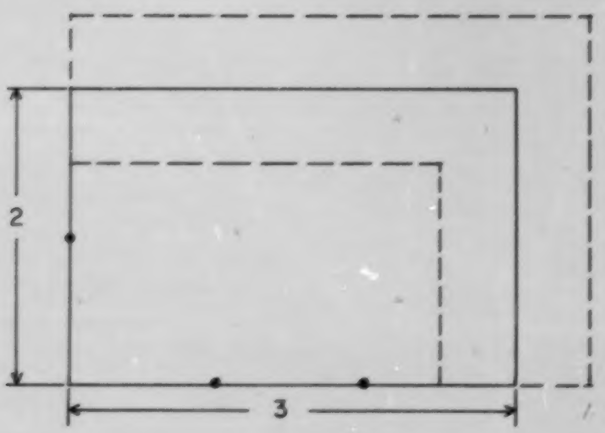
Consider the computations below.

Least values		Greatest values
17.5	18	18.5
15.5	16	16.5
271.25	288	305.25

In other words the "true" area can differ from 288 by as much as about 17 units. Being explicit, the best we could say is

288 ± 17.25

where actually 16.75 is the correct greatest possible error in the negative direction. The size of the greatest possible error has been massively magnified in the process of multiplication. We can see this geometrically by considering the figure below.



The 2 by 3 region is enclosed by the heavy segments. The 1 1/2 by 2 1/2 possibility is indicated on the inside and the 2 1/2 by 3 1/2 on the outside.

Going back to the 288 ± 17.25 case discussed above one might well ask, "How should the answer be written if we don't wish to indicate the greatest possible error explicitly?" There is no clear-cut answer. Some would prefer 288 but clearly this implies much greater precision than is present. Some would prefer 290. Here the "true" value would be indicated as being between 285 and 295 which, while not necessarily correct, seems not unreasonable. The figure 300 is far too imprecise for most purposes. On the basis of the three objectives for statements listed at the beginning of this chapter, it might be argued that 290 would be the best answer. The usual 288 seems too likely to be invalid. However, for most purposes the answer of 288 is used.

At this stage we can draw a distinction between what might be called "numerical fidelity" in arithmetic and preciseness of mathematical statements. When children multiply 8 by 7 they should get 56 every time. Any answer other than 56 is simply wrong. "Numerical fidelity" is important in arithmetic. But an answer of 56 sq. ft. for the area of a room 8 ft. by 7 ft. is justifiable primarily because we assume an answer is expected to the nearest square foot, and then 56 sq. ft. is the best we can do. The answer 56 sq. ft. is misleading in its implication of precision but other possible answers have their defects too. It is important to understand the limitations of our language and conventions.

We return to some examples involving computations. Let us ask how precise linear measurements should be in order for the product of the linear measurements to be precise to the nearest unit. Consider an example.

We want $10 \times 20 = 200$ to be precise to the nearest unit. Let t be the greatest possible error for each of 10 and 20. Then we have

Least values	Greatest values
$20 - t$	$20 + t$
$\underline{10 - t}$	$\underline{10 + t}$
$200 - 30t + t^2$	$200 + 30t + t^2$

Now if t is small then t^2 is much smaller. So let us consider only $30t$. Then $30t$ should be less than .5. In other words,

$$30t < \frac{1}{2} \quad \text{or} \quad t < \frac{1}{60} \qquad \text{or } t < .016.$$

Thus we see that in this case if $t < .016$ then the greatest possible error of the product is about .5. Our measurements 10 and 20 have to be very precise for the product to be reasonably precise.

Finally, let us make some observations about the relative error of a product. Here, in our example, things do better.

$$\frac{17.25}{288} \approx .06$$

$$\frac{.5}{16} \approx .03$$

$$\frac{.5}{18} \approx .03$$

We add the relative errors to get the relative error of the product. Let us justify this. Let N_1 and N_2 be the numbers to multiplied together. Let t_1 and t_2 be their respective greatest possible errors. Thus

$$\frac{t_1}{N_1} \quad \text{and} \quad \frac{t_2}{N_2}$$

are the original relative errors. Now

$$(N_1 + t_1) \cdot (N_2 + t_2) = N_1 \cdot N_2 + (t_1 \cdot N_2 + t_2 \cdot N_1) + t_1 \cdot t_2 \quad \text{and}$$

$$(N_1 - t_1) \cdot (N_2 - t_2) = N_1 \cdot N_2 - (t_1 \cdot N_2 + t_2 \cdot N_1) + t_1 t_2 .$$

If t_1 and t_2 are small, then $t_1 \cdot t_2$ is very small and we ignore it ($t_1 \cdot t_2$ was the .25 of our example). Hence $t_1 \cdot N_2 + t_2 \cdot N_1$ is (approximately) the greatest possible error in the product. Hence

$$\text{relative error} \approx \frac{t_1 \cdot N_2 + t_2 \cdot N_1}{N_1 \cdot N_2}$$

But $$\frac{t_1 \cdot N_2 + t_2 \cdot N_1}{N_1 \cdot N_2} = \frac{t_1 \cdot N_2}{N_1 \cdot N_2} + \frac{t_2 \cdot N_1}{N_1 \cdot N_2}$$

$$= \frac{t_1}{N_1} + \frac{t_2}{N_2} .$$

Thus

the relative error $\approx \frac{t_1}{N_1} + \frac{t_2}{N_2}$,

and the right hand side is the sum of the relative errors of the two factors.

Exercises 7-4

1. Find the greatest possible error and the relative error of the product of
   (a) 12 and 25.
   (b) .8 ± .01 and .6 ± .02.
2. Find (approximately) the greatest possible error of the factors 8 and 12 if the product is to have a
   (a) greatest possible error of .1.
   (b) relative error of .1.
   (Assume the two factors have the same greatest possible error.)
3. A house is advertised as 30 ft. by 36 ft. but each measurement is really almost 6 inches shorter than the figure given. The buyer thought he was getting 1080 square feet of house. How much was he actually getting?

4. What is the greatest possible error in the volume of a box given as 6" by 8" by 8"?
5. Find the approximate greatest possible error and relative error of the quotient of 35 divided by 7.

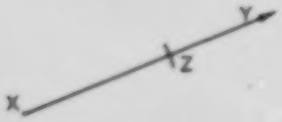
Chapter 8

Congruence

1. Informal Constructions.

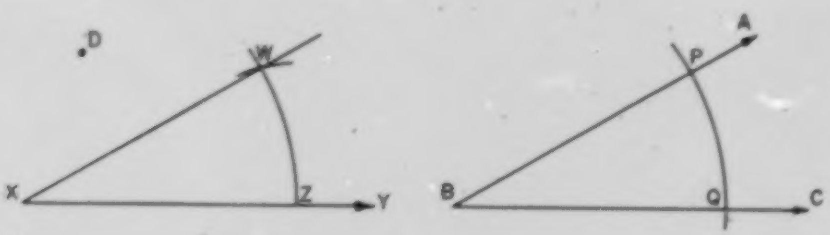
In Chapter 6 we have stated some basic properties about the existence of segments congruent to a given segment and of angles congruent to a given angle. In this section we discuss the geometric construction of such segments and angles and later of triangles. For these constructions, we assume we have available an unmarked ruler (a straight-edge) and a compass. These were the classical "tools" of the Greek geometers. If we wanted to make drawings or sketches as distinct from geometric constructions we could draw figures free-hand or use marked rulers and protractors. Here we limit ourselves to the classical "tools".

Segments. Given a segment $\overline{AB}$ and a ray $\overrightarrow{XY}$. How do we find a point Z on $\overrightarrow{XY}$ such that $\overline{AB} \cong \overline{XZ}$?



We can adjust the compass so that with the point at A the pencil tip will fall on B. Then with this setting we can put the point at X and mark an arc of a circle which crosses $\overrightarrow{XY}$. Call the point of intersection Z. Then $\overline{AB} \cong \overline{XZ}$. (We could also mark the straight edge--or note a marking on it--and use the marked straight edge to find the point Z.) Usually in geometry we prefer to use the compass for this construction whereas in measuring lengths in the everyday world we use the marked straight edge method.

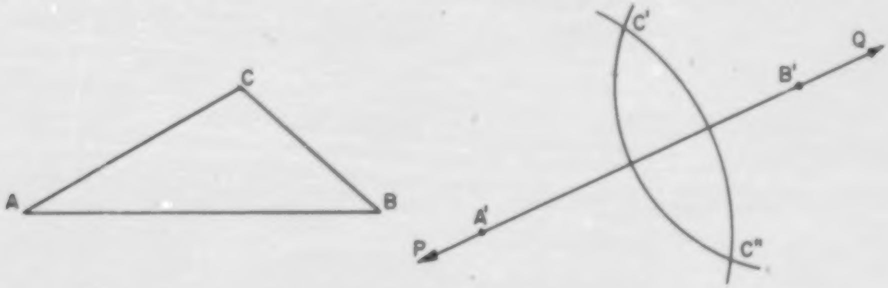
Angles. Given an angle $\angle ABC$, a ray $\overrightarrow{XY}$ and a point D not on the line $\overleftrightarrow{XY}$. How do we find a point W on the D-side of $\overleftrightarrow{XY}$ such that $\angle ABC \cong \angle WXY$?



We know such a point exists (from Property I-A of Chapter 6). The question is how do we use a ruler or compass (or both) to find it? With the compass point at B mark off an arc of a circle intersecting rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Call the points P and Q respectively. Mark off an arc of a circle as indicated with center at X, and with radius equal to BQ (or BP).

Now set the compass to measure the length of $\overline{PQ}$ (the segment $\overline{PQ}$ does not need to be drawn). With this setting and with the point of the compass at Z draw an arc intersecting the arc with center at X which has already been drawn. Finally if we call such a point of intersection W then the angle $\angle WXZ \cong \angle ABC$. At least, it looks as if $\angle WXZ$ should be congruent to $\angle ABC$. In Section 2 of this chapter we pin down the assumptions that let us assert such to be true.

Congruence of Triangles. (Informal). In traditional geometry, some of the principal theorems deal with congruence of triangles. We begin our study with some intuitive observations.

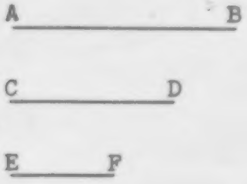


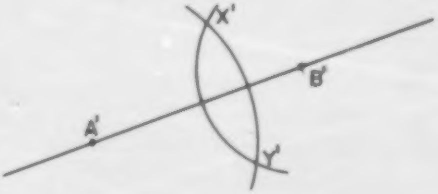
Suppose we have given $\triangle ABC$. How can we construct a triangle congruent to $\triangle ABC$ by use of a ruler and compass?

We lay off on the line $\overleftrightarrow{PQ}$ a segment $\overline{A'B'}$ which is congruent to $\overline{AB}$. (We put the point of the compass at any point A' and mark an arc of a circle crossing $\overleftrightarrow{PQ}$ at a point we call B'.) With A' as center we draw a circle (or an arc of a circle) with radius equal to the length of $\overline{AC}$. With B' as center we draw a circle (or an arc of a circle) with radius equal to the length of $\overline{BC}$. The two circles we are considering intersect in two points. Call these points C' and C''. They will be on opposite sides of the line $\overleftrightarrow{PQ}$. Then $\triangle ABC \cong \triangle A'B'C'$ and also $\triangle ABC \cong \triangle A'B'C''$. If we were to try to superimpose $\triangle ABC$ on $\triangle A'B'C'$, for instance, everything would fit.

We could begin this way. Lay $\overline{AB}$ on A'B' with A on A' (and hence B on B'). Then C would have to fall on the circle with center at A' and radius the length of $\overline{AC}$. Also C would have to fall on the circle with center at B' and radius equal to the length of $\overline{BC}$. Therefore C would have to fall on the point C' (or the point C'') as these are the only two points on both circles. Now we could require that C fall on the C' side of $\overleftrightarrow{PQ}$ and thus C must fall on C'. Therefore our congruence seems to be established. Similarly $\triangle ABC \cong \triangle A'B'C''$. Thus using a ruler and compass we have seen how to construct a copy of a triangle.

Let us consider a similar problem. Suppose we are given three segments as follows.





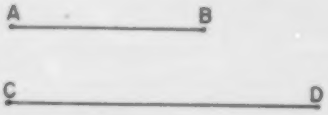
Construct a triangle whose sides are congruent to $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$. The construction would go through like the one above. We would lay off $\overline{A'B'}$ with $AB = A'B'$. We would construct a circle with center at A' and with radius the length of $\overline{CD}$. We would construct a circle with center at B' and with radius the length of $\overline{EF}$. Then if the two circles intersect in two points, say X' and Y', either X' or Y' may be taken as the third vertex of the desired triangle.

It is interesting to note what would happen if $AB = CD + EF$ [or in the other notation if $m(\overline{AB}) = m(\overline{CD}) + m(\overline{EF})$]. In this case the two circles would intersect in just one point (the point of tangency) and that point would be on $A'B'$. Hence no triangle could be formed.

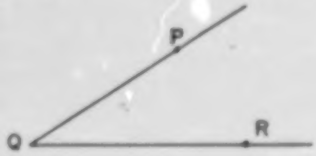
Finally if $AB > (CD + EF)$ then the intersection of the two circles would be the empty set and again no triangle could be formed. In Chapter 9 we shall note such a relationship again, the so-called triangle inequality. In any triangle, the length of any side is less than the sum of the lengths of the other two.

Exercises 8-1

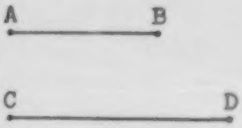
1. Given segments $\overline{AB}$ and $\overline{CD}$ below. Draw a ray. Label it $\overrightarrow{PQ}$. With a compass find points X and Y of $\overrightarrow{PQ}$ such that $\overline{PX} \cong \overline{AB}$ and $\overline{XY} \cong \overline{CD}$.

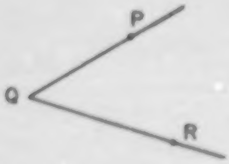


2. Given angle $\angle PQR$ below. Draw a ray $\overrightarrow{XY}$ and a point Z not on $\overleftrightarrow{XY}$. Construct an angle $\angle WXY$ such that W is on the Z-side of $\overleftrightarrow{XY}$ and $\angle PQR \cong \angle WXY$.



3. (a) Suppose we have given two segments and one angle.



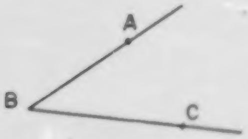


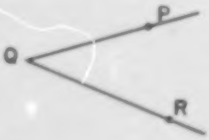
Construct a triangle with two sides congruent to $\overline{AB}$ and and $\overline{CD}$ and with the angle included between these sides congruent to $\angle PQR$.

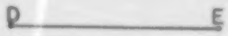
(b) Once the angle and two sides (with the angle between them) are known is the triangle completely determined?

(c) Can anybody give two segments and an angle for which this construction is impossible? Explain.

4. (a) Suppose we have given two angles and one segment





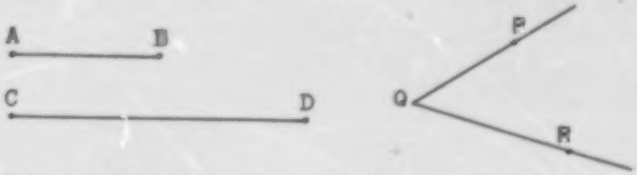


Construct a triangle with a side congruent to $\overline{DE}$ and the two angles adjacent to such side congruent to $\angle ABC$ and $\angle PQR$.

(b) Once the two angles and the side between them are known is the triangle completely determined?

(c) Can anybody give two angles and a segment for which this construction is impossible? Explain.

5. (a) Suppose we have given two segments and one angle



Construct a triangle with two sides congruent to $\overline{AB}$ and $\overline{CD}$ and with an angle not included between them congruent to $\angle PQR$. Require this angle to be adjacent to the side congruent to $\overline{AB}$.

(b) The same as (a) except require this angle to be adjacent to the side congruent to $\overline{CD}$. Is the construction possible?

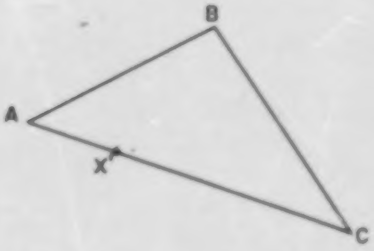
(c) If $\overline{AB}$ were enough longer could the construction of (b) be done in two different ways?

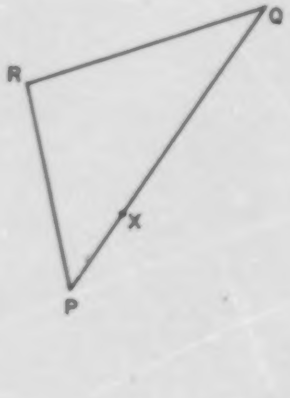
(d) Can anybody give two segments and an angle for which neither the construction of (a) nor that of (b) is possible?

2. The Meaning of Congruence.

In this section, we try to give a more explicit definition of congruence and to show the relationship of this definition to previous understandings. We have said that two sets of points are congruent if they have the "same size and shape". In traditional terminology, this is interpreted as meaning "if either figure (set of points) can be superimposed on the other". But as we have remarked in Chapter 6, the process of superposition gets us involved with considerations of "moving objects around", and, from some points of view, the motion involved is irrelevant to the idea of congruence. Also while we shall be primarily concerned with congruence between sets of points in a plane, the definition we use is applicable to sets of points in space. The idea of superimposing one billiard ball on another doesn't make much sense. Yet billiard balls are "congruent". The definition we give should help pin-point the basic idea of congruence and emphasize its applicability to various types of figures.

If we look at it in a certain way, the idea of superimposing one figure on another leads us directly to our definition of congruence.





Suppose $\triangle PRQ$ can be superimposed on $\triangle ABC$ with R falling on B, P on A and Q on C. Then there exists a 1-1 correspondence between $\triangle PRQ$ and $\triangle ABC$, each point of $\triangle PRQ$ corresponding to that point of $\triangle ABC$ which it "covers" when $\triangle PQR$ is superimposed on $\triangle ABC$. For example, the point X would correspond to the point X' under this correspondence. But it is not enough simply to say that there exists a 1-1 correspondence between $\triangle PRQ$ and $\triangle ABC$. Something else is also involved in the notion of congruence. *Distances must be preserved*. Suppose $\triangle PRQ$ is superimposed on $\triangle ABC$ as indicated

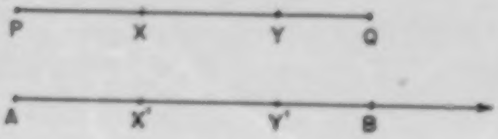
$$\begin{aligned} P &\longleftrightarrow A \\ R &\longleftrightarrow B \\ Q &\longleftrightarrow C \end{aligned}$$

then for *any* two points of $\triangle PRQ$, the distance between them, (i.e., the length of the segment joining them) must be the same as the distance between the two points of $\triangle ABC$ which they cover, i.e., between the two points of $\triangle ABC$ which they correspond to under the 1-1 correspondence. As examples, the distance between R and X must be the same as the distance between B and X' (in other words, $RX = BX'$), the distance between Q and P must be the same as that between C and A ($QP = CA$), and the distance between Q and X must be the same as that between C and X' ($QX = CX'$).

These considerations lead us now to our definition:

Definition: *Two sets of points are said to be congruent provided that there is a one-to-one correspondence between them which preserves distance.*

With this definition in mind let us go back to considerations of congruence between two segments and congruence between two angles.

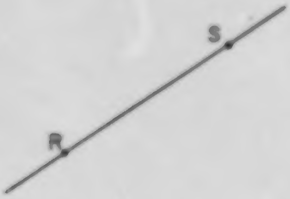


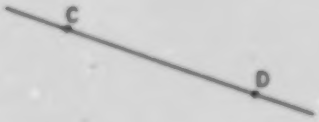
Saying that $\overline{AB}$ is congruent to $\overline{PQ}$ means that there is a one-to-one correspondence between $\overline{AB}$ and $\overline{PQ}$ (as sets of points) and distance is preserved under this correspondence. If we think about laying off $\overline{PQ}$ on the ray $\overline{AB}$ as suggested by the drawing above then $P \longleftrightarrow A$, $Q \longleftrightarrow B$ and for any point X of $\overline{PQ}$ there is a corresponding point X' of $\overline{AB}$. Furthermore, distance is preserved. For example, if $X \longleftrightarrow X'$, and $Y \longleftrightarrow Y'$, the length of $\overline{XY}$ is equal to the length of $\overline{X'Y'}$. A statement of the existence of a congruence should be understood to imply the existence of a one-to-one correspondence which preserves distance.

It is hard to check on whether all distances between pairs of corresponding points are preserved. We want conditions which we can observe and which tell us that such a one-to-one correspondence which preserves distance must exist.

It is really part of our basic understanding about congruence of segments that the following property holds. We understand $\{A, B\}$ to mean the set consisting of the two elements A and B.

Property I: *If* R, S, C, *and* D *are points and* $[R, S] \cong [C, D]$, *then* $\overline{RS} \cong \overline{CD}$. *Furthermore, there are exactly two congruences of* $\overline{RS}$ *with* $\overline{CD}$ *which corresponds* $[R, S]$ *with* $[C, D]$. *One of these corresponds* R *with* C, *the other* R *with* D.

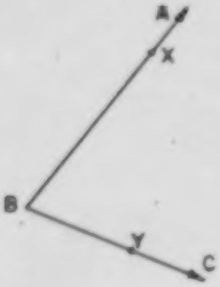


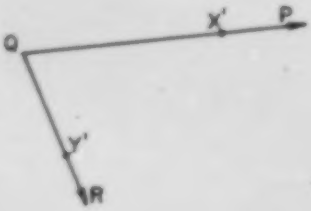


One thing this property says is that all segments of a given length are alike. Any congruence between two pairs of points induces a unique congruence between the two segments having these pairs of points as endpoints. In fact, if $[R, S] \cong [C, D]$ and $R \longleftrightarrow C$, $S \longleftrightarrow D$ then there is a unique one-to-one correspondence between $\overline{RS}$ and $\overline{CD}$ which preserves distance and corresponds R with C and S with D.

There is one and only one way of laying segment $\overline{RS}$ on segment $\overline{CD}$ so that $R \longleftrightarrow C$ and $S \longleftrightarrow D$.

We now consider the congruence of two angles.





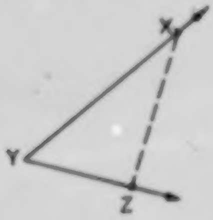
The angles $\angle PQR$ and $\angle ABC$ are congruent if either can be superimposed on the other or, more precisely, if there exists a one-to-one correspondence between them which preserves distance. As, under these conditions, the vertices B and Q must correspond to each other, then ray $\overrightarrow{BA}$ can be identified with ray $\overrightarrow{QP}$ or with ray $\overrightarrow{QR}$. Either of these leads to a congruence of the two angles. Let us suppose ray $\overrightarrow{BA}$ is identified with ray $\overrightarrow{QP}$. Then any point X of $\overrightarrow{BA}$ corresponds to a point X' of $\overrightarrow{QP}$ and any point Y of $\overrightarrow{BC}$ to a point Y' of $\overrightarrow{QR}$. (In the case of angles, the points A, C, P and R that we used to name the angles may not correspond to each other.)

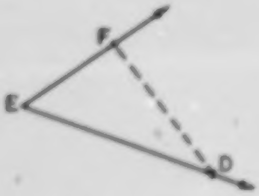
The implicit assumption about the congruence of $\angle ABC$ with $\angle PQR$ is that distances will be preserved under the one-to-one correspondence which is set up. Thus in our figure $\overline{BX} \cong \overline{QX'}$, $\overline{BY} \cong \overline{QY'}$, and $\overline{XY} \cong \overline{X'Y'}$. The last of these is important to note. The distance between any pair of points of $\angle ABC$ is equal to the distance between the pair of corresponding points of $\angle PQR$. In effect we assumed this to be so when we first gave our properties on congruence of angles.

In the case of segments, two segments were congruent if their two sets of endpoints were congruent. A similar type of condition is true for angles. We wish to state explicitly our basic understanding.

Property II: *Consider* $\angle XYZ$ *and* $\angle DEF$ *such that* $\overline{YX} \cong \overline{ED}$, $\overline{YZ} \cong \overline{EF}$, *and* $\overline{XZ} \cong \overline{DF}$. *Then* $\angle XYZ \cong \angle DEF$ *and with* $X \leftrightarrow D$, $Y \leftrightarrow E$, *and* $Z \leftrightarrow F$ *there is a unique such congruence.*

This property tells us that if we can find three points of either angle in the correct relationship to some three points of the other, then the angles are congruent.



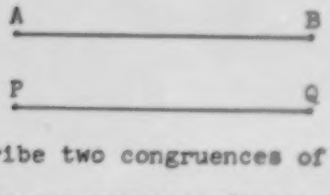


This property gives us a criterion for stating that two angles are congruent. It is precisely this type of condition that we needed in Section 1 of this chapter to assert that our construction actually gave an angle congruent to the given one.

Property II is really rather intuitive. We would expect $\angle DEF$ to coincide with $\angle XYZ$ if we superimposed the figures with F on Z, D on X and E on Y.

Exercises 8-2

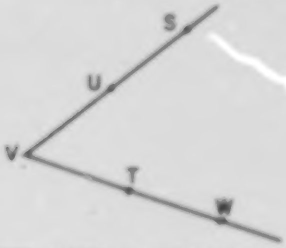
1. $\overline{AB}$ and $\overline{PQ}$ are given below as having the same length.

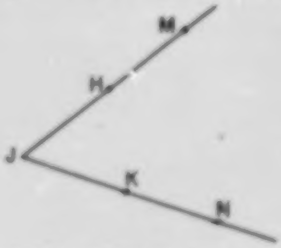


Describe two congruences of $\overline{AB}$ and $\overline{PQ}$, i.e., describe two one-to-one correspondences between $\overline{AB}$ and $\overline{PQ}$ which preserve distance.

2. In (1) above suppose X is a point of $\overline{AB}$ at $\frac{1}{4}$ of the distance from A to B. Draw a copy of $\overline{PQ}$ and label as Y and Y' the points to which X would correspond under the two congruences of (1).

3. Assume that the angles $\angle UVW$ and $\angle HJK$ below are congruent.



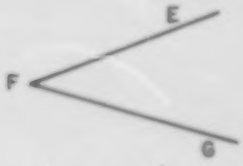


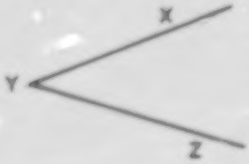
Assume $\overline{VU} \cong \overline{VT} \cong \overline{HJ} \cong \overline{JK}$ and $\overline{VS} \cong \overline{VW} \cong \overline{JM} \cong \overline{JN}$.

(a) Describe two congruences of $\angle UVW$ with $\angle HJK$ by matching the five indicated points of one figure with the five indicated points of the other in two different ways.

(b) In one of your congruences of (a) $U \longleftrightarrow H$ and $W \longleftrightarrow N$. What do we know about $\overline{UW}$ and $\overline{HN}$? About $\overline{SW}$ and $\overline{MN}$?

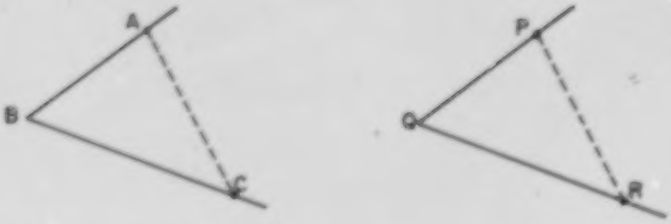
4. Consider the figures below.





Explain how by measuring three segments of each figure we might prove that $\angle EFG$ is congruent to $\angle XYZ$ (if it is).

5. Consider the angles below.



Suppose $\overline{AB} \cong \overline{QP}$ and $\overline{BC} \cong \overline{QR}$.

If $\overline{AC}$ is not congruent to $\overline{PR}$, can $\angle ABC$ be congruent to $\angle PQR$? Explain.

6. Try to state Property II more simply.

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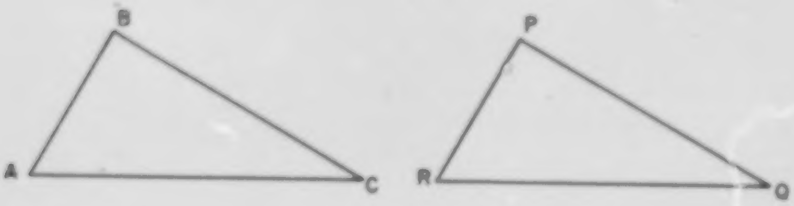
3. Congruence of Triangles.

What is usually meant by saying that $\triangle ABC \cong \triangle RPQ$? In traditional terminology one says that $\triangle ABC$ can be superimposed on $\triangle RPQ$. In many geometry texts this is also taken to mean that

> "corresponding sides are equal and corresponding angles are equal".



Of course, in our terminology the sides and angles are sets of points and, hence the word "equal" would be replaced by "congruent". Note that both of the above meanings for congruence of triangles involve a matching process or correspondence.

Certainly the superposition requires a one-to-one correspondence between the two sets, each point of the one set corresponding to the point of the other on which it is superimposed. If $\overline{AB}$ corresponds to $\overline{RP}$ and $\overline{BC}$ corresponds to $\overline{PQ}$ then clearly $B = \overline{AB} \cap \overline{BC}$ should correspond to $P = \overline{RP} \cap \overline{PQ}$. Thus the idea of "corresponding sides" and "corresponding angles" requires that the set of vertices of the one triangle be in a particular one-to-one correspondence with the set of vertices of the other. In fact, the converse is also true; a particular one-to-one correspondence of the two sets of vertices induces (or produces) a one-to-one correspondence between the sets of sides of the two triangles and a similar correspondence between the sets of angles of the two triangles. For instance, if $A \longleftrightarrow R$ and $B \longleftrightarrow P$ then $\overline{AB} \longleftrightarrow \overline{RP}$.

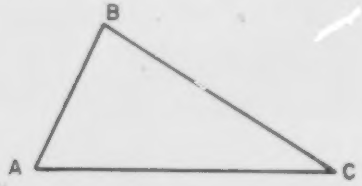
Thus we see that a key to the possible congruence of two triangles is a matching of their sets of vertices. In fact, we have the following almost obvious theorem which we give without proof.

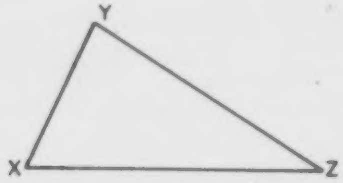
Theorem I: If $\triangle ABC \cong \triangle PQR$, then any one-to-one correspondence of the triangles which preserves distance gives a one-to-one distance-preserving correspondence of the sets of vertices of the two triangles (of $\{A, B, C\}$ with $\{P, Q, R\}$).

To make our notation and language easier, let us agree that writing

$$\triangle ABC \cong \triangle XYZ$$

means not only that the triangles are congruent under some matching process but that they are congruent under a one-to-one correspondence which matches the vertices in the order given. In other words, if we write $\triangle ABC \cong \triangle XYZ$ then we imply that





$A \longleftrightarrow X$, $B \longleftrightarrow Y$, and $C \longleftrightarrow Z$ under the congruence we have in mind. Similarly, let us agree that $\{A, B, C\} \cong \{P, Q, R\}$ implies that $A \longleftrightarrow P$, $B \longleftrightarrow Q$, and $C \longleftrightarrow R$ under the congruence implied between the two sets of three points each.

The converse of Theorem I, which we shall state as Theorem II, is also true but it requires some proof which we shall outline. (We assume in Theorem II that $\{A, B, C\}$ and $\{P, Q, R\}$ are sets of vertices of triangles.)

Theorem II: If $\{A, B, C\} \cong \{Q, P, R\}$, then $\triangle ABC \cong \triangle QPR$.

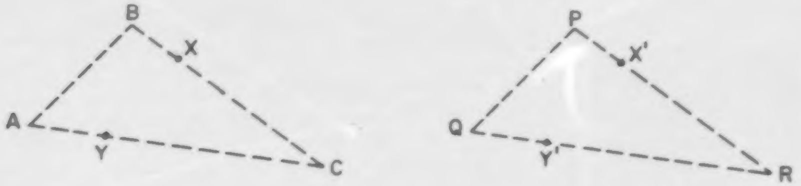
Proof: We begin by recalling what we mean by saying that $\{A, B, C\} \cong \{Q, P, R\}$. A set of points is congruent to another if there is a one-to-one correspondence which preserves distance between them. Therefore, as $A \longleftrightarrow Q$, $B \longleftrightarrow P$, and $C \longleftrightarrow R$, we are saying that $AB = QP$, $AC = QR$, and $BC = PR$, these indicating the distances that must be preserved.

But Property I of the previous section implies that there are one-to-one correspondences which preserve distance between

$\overline{AB}$ and $\overline{QP}$ with $A \longleftrightarrow Q$ and $B \longleftrightarrow P$
$\overline{AC}$ and $\overline{QR}$ with $A \longleftrightarrow Q$ and $C \longleftrightarrow R$
and $\overline{BC}$ and $\overline{PR}$ with $B \longleftrightarrow P$ and $C \longleftrightarrow R$

Thus we may consider a one-to-one correspondence to have been set up between $\triangle ABC$ and $\triangle QPR$ and this correspondence matches A with Q, B with P and C with R.

What we have not yet observed is whether or not all distances are preserved under the correspondence. For instance, in the figure below, if $X \longleftrightarrow X'$ and $Y \longleftrightarrow Y'$, is $AX = QX'$? Is $XY = X'Y'$?



The answer to each of these questions is "yes" and we use Property II of the previous section to see that such should be the case.

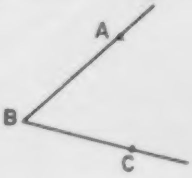
We note that $\overline{AC} \cong \overline{QR}$, $\overline{BC} \cong \overline{PR}$ and $\overline{AB} \cong \overline{QP}$ and these are just what we need to apply Property II. From Property II, then, $\angle ACB \cong \angle QRP$ with $A \longleftrightarrow Q$, $C \longleftrightarrow R$ and $B \longleftrightarrow P$ and the "natural" further correspondence such as $X \longleftrightarrow X'$ and $Y \longleftrightarrow Y'$. But as $A \longleftrightarrow Q$, $X \longleftrightarrow X'$ and $Y \longleftrightarrow Y'$ then AX must be equal to QX' and XY must be equal to X'Y' because distance must be preserved under congruence of the angles $\angle ACB$ and $\angle QRP$, in this instance. By use of this type of reasoning Theorem II can be established on the basis of our assumed properties.

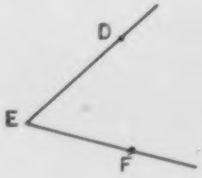
Note that in this argument which we have sketched, we *first* observed that the corresponding angles were congruent (using Property II). Then because corresponding angles were congruent the various distances had to be preserved.

We now wish to observe the fundamental theorem that if two triangles are congruent by our definition, then they are by the traditional definition.

Theorem III: *If* $\triangle ABC \cong \triangle DEF$, *then the corresponding sides and corresponding angles of the two triangles are congruent*.

Proof: Since $A \leftrightarrow D$ and $B \leftrightarrow E$ under our congruence, then $AB = DE$. Therefore $\overline{AB} \cong \overline{DE}$. Similarly $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$, and the corresponding sides are congruent. What about $\angle ABC$ and $\angle DEF$? Are they congruent? The answer is yes, for





$\overline{BA} \cong \overline{ED}$, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$ as we have observed. Then by Property II, $\angle ABC \cong \angle DEF$ under a correspondence for which $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. Similarly $\angle BAC \cong \angle EDF$ and $\angle BCA \cong \angle EFD$.

Thus we see that in the case of triangles our definition of congruence implies the traditional one. Why then do we use it?

(1) It is more explicit and leads to better understanding.

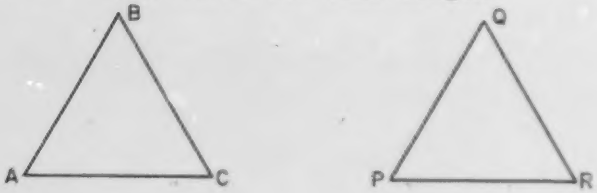
(2) It emphasizes the fundamental idea of congruence and in so doing is applicable to other types of figures (sets of points).

(3) It does not unnecessarily introduce the idea of "moving" sets.

(4) It gives another elementary geometric setting to illustrate the important idea of a one-to-one correspondence. Thus it helps give a unity to the language of mathematics.

Exercises 8-3

1. Suppose $\triangle ABC$ and $\triangle PQR$ are as in the figures below with all six indicated segments of the same length.



How many congruences are there between $\triangle ABC$ and $\triangle PQR$? List the matching of the sets of vertices for all of them. For example, $(A \longleftrightarrow P,\ B \longleftrightarrow R,\ C \longleftrightarrow Q)$ would be one such.

2. Explain why Theorem II is like the traditional side-side-side congruence theorem.

3. Suppose in triangles $\triangle DEF$ and $\triangle XYZ$ that

   $\overline{DE}$ is not congruent to $\overline{XY}$

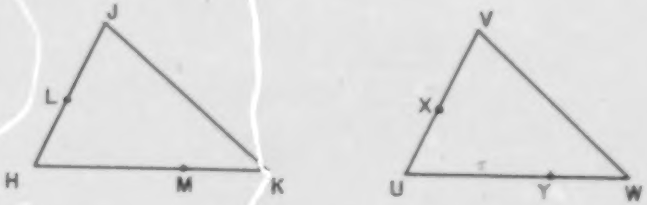
   $\overline{DF}$ is not congruent to $\overline{XZ}$

   and $\overline{EF}$ is not congruent to $\overline{YZ}$.

   Can $\triangle DEF$ be congruent to $\triangle XYZ$?

   Must the two triangles be congruent? Explain.

4. Suppose $\triangle HJK \cong \triangle UVW$ as below, with $L \longleftrightarrow X$, $M \longleftrightarrow Y$.



   List all the pairs of segments (with some of the indicated five points as endpoints) which you know must be of equal length. You should have 10 of them.

5. Suppose $\triangle PQR \cong \triangle ABC$.

   Explain how we know that $\angle PQR \cong \angle ABC$.

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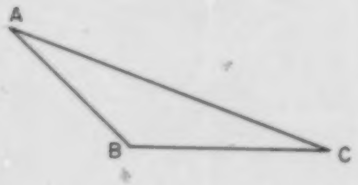
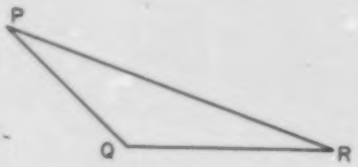
4. Congruence of Triangles--The Standard Theorems.

We begin with the SSS Theorem (Side-Side-Side).

Theorem IV: Consider $\triangle ABC$ and $\triangle PQR$. If $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, and $\overline{AC} \cong \overline{PR}$ then $\triangle ABC \cong \triangle PQR$.

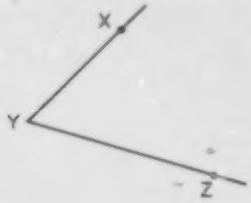


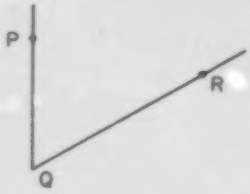


Proof: This theorem is essentially a restatement of Theorem II. Let $A \leftrightarrow P$, $B \leftrightarrow Q$ and $C \leftrightarrow R$. Then this correspondence is a congruence of $\{A, B, C\}$ with $\{P, Q, R\}$. Therefore Theorem II asserts that $\triangle ABC \cong \triangle PQR$.

Next we state the S∠S Theorem (side-angle-side).

Theorem V: Consider $\triangle XYZ$ and $\triangle PQR$. If $\overline{XY} \cong \overline{PQ}$, $\overline{YZ} \cong \overline{QR}$ and $\angle XYZ \cong \angle PQR$, then $\triangle XYZ \cong \triangle PQR$.



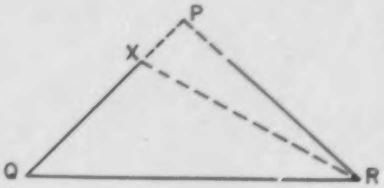


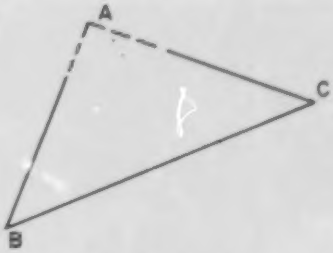
Proof: The given condition that the angles $\angle XYZ$ and $\angle PQR$ are congruent means that there is a one-to-one distance-preserving correspondence between the angles. This correspondence can be taken so that $X \leftrightarrow P$, $Y \leftrightarrow Q$ and $Z \leftrightarrow R$. But since all corresponding distances must be equal, $XZ = PR$ and then $\overline{XZ} \cong \overline{PR}$.

This together with the given congruences of segments asserts conditions like those of the hypotheses of Theorem IV. Therefore $\triangle XYZ \cong \triangle PQR$.

We now consider the $\angle S \angle$ Theorem (angle-side-angle).

Theorem VI: *Consider triangles* $\triangle PQR$ *and* $\triangle ABC$. *If* $\angle PQR \cong \angle ABC$, $\overline{QR} \cong \overline{BC}$, *and* $\angle QRP \cong \angle BCA$, *then* $\triangle PQR \cong \triangle ABC$.

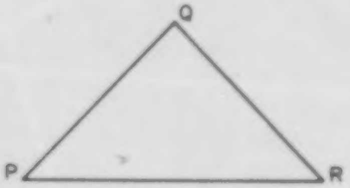




Proof: Let X be a point on $\overrightarrow{QP}$ such that $\overline{QX} \cong \overline{BA}$ and X is on the P-side of $\overleftrightarrow{QR}$. From Theorem V, $\triangle XQR \cong \triangle ABC$. Thus by Theorem III, $\angle XRQ \cong \angle ACB$. But by Property I-A of Chapter 6, there is only one ray with endpoint at R and containing a point on the P-side of $\overleftrightarrow{QR}$ such that the angle formed by this ray and $\overrightarrow{RQ}$ is congruent to $\angle ACB$. Therefore X and P must both be on this ray and hence on the line $\overleftrightarrow{PR}$. But X and P are both on the line $\overleftrightarrow{QP}$. These two lines can have at most one point of intersection. Therefore the point X is the point P and $\overline{QP} \cong \overline{BA}$. Now the conditions for Theorem V are obtained and hence the two triangles are congruent.

A triangle is called *equilateral* if its three sides are all congruent to each other. A triangle is called *isosceles* if some two sides of it are congruent to each other. We list two of the most fundamental theorems about isosceles triangles. These theorems are used to prove various other theorems.

Theorem VII: *If two sides of a triangle are congruent, then the angles opposite these sides are congruent.*



We are given that $\overline{PQ} \cong \overline{QR}$. We wish to show that $\angle QPR \cong \angle QRP$.

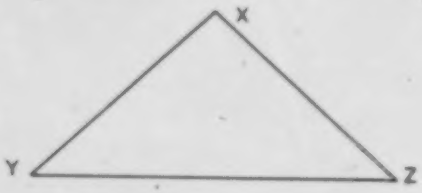
Proof: We note that $\{P, Q, R\} \cong \{R, Q, P\}$, for

$$\overline{PQ} \cong \overline{RQ}$$
$$\overline{PR} \cong \overline{PR} \text{ (or } \overline{RP}\text{)}$$
$$\overline{QR} \cong \overline{QP}.$$

Therefore, by Theorem II, $\triangle PQR \cong \triangle RQP$. But $\angle QPR$ corresponds to $\angle QRP$ under this congruence and thus, by Theorem III, $\angle QPR \cong \angle QRP$, as was to be shown.

Finally we state the converse of Theorem VII.

Theorem VIII: *If two angles of a triangle are congruent, then the sides opposite these angles are congruent.*



We are given that $\angle XYZ \cong \angle XZY$. We wish to prove that $\overline{XZ} \cong \overline{XY}$.

Proof: Consider the correspondence as follows:

$X \longleftrightarrow X$

$Y \longleftrightarrow Z$

$Z \longleftrightarrow Y$

Under this correspondence of vertices

$\angle XYZ$ corresponds to $\angle XZY$

$\overline{YZ}$ corresponds to $\overline{ZY}$

$\angle XZY$ corresponds to $\angle XYZ$.

But it is given that each angle cited is congruent to its corresponding angle. Also $\overline{YZ} \cong \overline{ZY}$ by identity. Therefore the conditions of Theorem VI are achieved. Hence $\triangle YXZ \cong \triangle ZXY$. Under this congruence $\overline{XY} \longleftrightarrow \overline{XZ}$ and thus, by Theorem III, $\overline{XY} \cong \overline{XZ}$ as was to be shown.

Exercises 8-4

1. Prove that all the angles of an equilateral triangle are congruent to each other.
2. Draw figures to show that the side-side-angle "theorem" is not true. In other words, exhibit two triangles which are not congruent, but for which two sides and a non-included angle of the one are congruent respectively to two sides and a non-included angle of the other.
3. Give examples of two equilateral triangles which are not congruent to each other. Hence show that the angle-angle-angle "theorem" is not true.

4. Given a quadrilateral ABCD whose opposite sides are congruent, i.e., $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$.

   Prove $\angle BAD \cong \angle BCD$ and

   $\angle ABC \cong \angle ADC$.

5. Given quadrilateral PQRS with $\overline{PQ} \cong \overline{PS}$ and $\overline{QR} \cong \overline{RS}$.

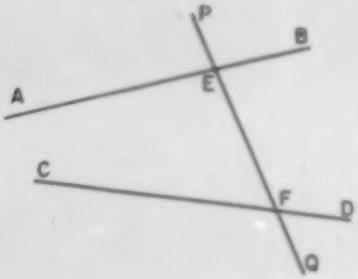
   Prove $\angle PQR \cong \angle PSR$.

Chapter 9

Parallels and Metric Properties of Triangles

1. *Terminology and Basic Properties.*

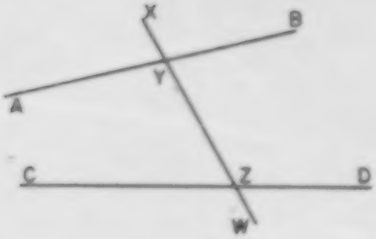
Suppose $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines in a plane and $\overleftrightarrow{PQ}$ is a line different from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Suppose $\overleftrightarrow{PQ}$ intersects both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ (in non-empty intersections). Then $\overleftrightarrow{PQ}$ is called a *transversal* of the two lines.



Let the points of intersection be E and F as in the figure. Assuming the various points are located as above we call $\angle PEB$ and $\angle EFD$ *corresponding angles*. There are three other pairs of corresponding angles in our figure. Similar ones are noted in the proof of Theorem I below.

We call $\angle AEF$ and $\angle EFD$ *alternate interior angles*. There is one other pair of alternate interior angles in our figure.

Theorem I: *If two corresponding angles are congruent to each other, then so are the angles of the other three pairs of corresponding angles.*



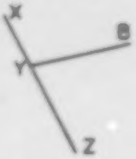
Consider the figure on the right and suppose $\angle XYB \cong \angle YZD$. We wish to show that

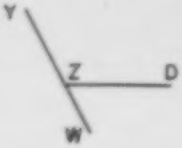
$\angle BYZ \cong \angle DZW$

$\angle AYX \cong \angle CZY$ and

$\angle AYZ \cong \angle CZW$

Proof: $\angle BYZ \cong \angle DZW$, supplements of angles given as congruent.





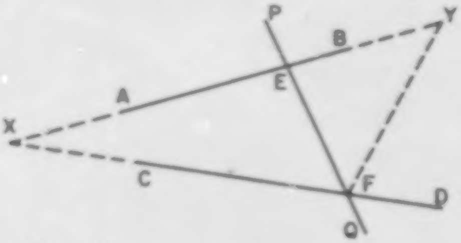
Similarly $\angle AYX \cong \angle CZY$, supplements of angles given as congruent.

Finally $\angle AYZ \cong \angle CZW$, supplements of angles proved to be congruent.

We next wish to establish a basic theorem about corresponding angles. We are given a tranversal cutting two lines.

Theorem II: If a pair of corresponding angles are congruent to each other, then the lines cut by the transversal are parallel.

We shall prove this theorem by contradiction. (Some readers may wish to use drawings of their own while reading this argument.)



We are given lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ cut by transversal $\overleftrightarrow{PQ}$ as in the figure. We are further given that $\angle PEB \cong \angle EFD$ (even if it doesn't look like it). Suppose X is an element of $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$. Let Y be a point on the ray $\overrightarrow{EB}$ such that $\overline{EY} \cong \overline{XF}$. Thus $Y \neq X$ and Y is on the D-side of line $\overleftrightarrow{PQ}$. Now consider $\triangle XEF$ and $\triangle EYF$. Let

$$X \longleftrightarrow Y$$
$$F \longleftrightarrow E$$
$$E \longleftrightarrow F$$

where the first listed points are thought of as the vertices of $\triangle XEF$.

Now $\overline{XF} \cong \overline{YE}$ by construction (i.e., defining condition for Y)

$\overline{EF} \cong \overline{FE}$ by identity

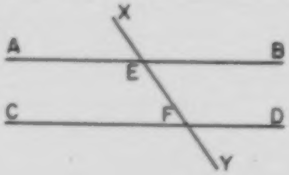
$\angle EFX \cong \angle FEY$ because supplements of congruent angles are congruent.

$\therefore$ $\triangle XEF \cong \triangle YFE$ by the SLS theorem with the correspondence between sets of vertices as above. Hence $\angle EFY \cong \angle XEF$ (corresponding angles of congruent triangles). But $\angle PEB \cong \angle XEF$ (vertical angles) and thus $\angle PEB \cong \angle EFY$. Also $\angle PEB \cong \angle EFD$ (given) and Y and D are on the same side of $\overleftrightarrow{PQ}$. Therefore $\angle EFY = \angle EFD$ (i.e., they are the same angle) by Property I-A of Chapter 6, which says there is a unique angle congruent to $\angle PEB$ with one ray $\overrightarrow{FP}$ and the other containing points on the B-side of $\overleftrightarrow{PQ}$.

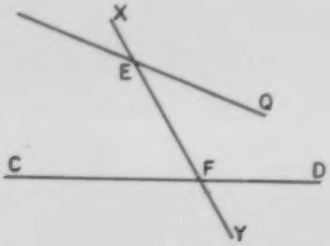
Thus Y must be on the line $\overleftrightarrow{CD}$. Therefore, line $\overleftrightarrow{AB}$ and line $\overleftrightarrow{CD}$ have the two points X and Y in common which is a contradiction. (Two distinct lines can have at most one point in common--Property I-A of Chapter 5.) Hence, the assumption that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have a non-empty intersection is false. Therefore the lines are parallel.

We wish to establish the converse of Theorem II.

Theorem III: If two parallel lines are cut by a transversal, then the corresponding angles are congruent. Given parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and transversal $\overleftrightarrow{XY}$ as in the figure.



We wish to prove that $\angle XEB \cong \angle EFD$. (Then, by Theorem I, the angles of all pairs of corresponding angles are congruent). There must exist a ray $\overrightarrow{EQ}$ such that $\angle XEQ \cong \angle EFD$ and Q is on the B-side of $\overleftrightarrow{XY}$. By Theorem II, $\overleftrightarrow{EQ}$ must be parallel to $\overleftrightarrow{FD}$. But there is only one line through E parallel to $\overleftrightarrow{FD}$ (Property V of Chapter 5). Hence $\overleftrightarrow{EQ}$ is $\overleftrightarrow{AB}$ ($\overleftrightarrow{EQ} = \overleftrightarrow{AB}$) and $\angle XEQ$ is $\angle XEB$. Therefore $\angle XEB \cong \angle EFD$ as was to be shown.



Exercises 9-1

1. (a) Prove that if a pair of corresponding angles are congruent, then so is some pair of alternate interior angles.

   (b) Prove the converse of (a).

2. Prove Theorem I-A: If two alternate interior angles are congruent to each other, then so are the angles of the other pairs of alternate interior angles.

3. Prove Theorem III-A. If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.
4. Try to simplify the proof of Theorem II.

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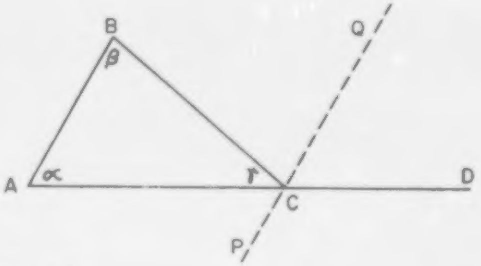
2. The Sum of the Measures of the Angles of a Triangle.

In this section we prove the following well-known theorem.

Theorem IV: If α, β, and γ (alpha, beta, and gamma) are the (degree) measures of the three angles of a triangle, then $\alpha + \beta + \gamma = 180$.

Given

$\triangle ABC$ with $\alpha = m(\angle BAC)$,
$\beta = m(\angle ABC)$ and
$\gamma = m(\angle ACB)$.



We wish to prove that $\alpha + \beta + \gamma = 180$.

Let $\overleftrightarrow{PQ}$ be the line through C which is parallel to $\overleftrightarrow{AB}$. We may regard Q as on the B-side of $\overleftrightarrow{AD}$ and in fact, in the interior of $\angle BCD$. Thus, $\angle QCD \cong \angle BAC$ (corresponding angles) and hence $\alpha = m(\angle QCD)$, for congruent angles have equal measure. Also $\angle BCQ \cong \angle ABC$ (alternate interior angles--see Exercise 3 of Section 1) and hence $\beta = m(\angle BCQ)$. Now $m(\angle BCD) = m(\angle QCD) + m(\angle BCQ) = \alpha + \beta$. But $m(\angle BCD) + m(\angle BCA) = 180$ (supplementary angles) and $(\alpha + \beta) + \gamma = 180$ or $\alpha + \beta + \gamma = 180$ as was to be shown.

We speak of $\angle BCD$ as an exterior angle of $\triangle ABC$. (Note that an exterior angle of a triangle is not a part of the triangle. It won't be by almost any definition which is used. Yet it has long been customary to use the expression "an exterior angle of a triangle".)

We have shown in the preceding proof that $m(\angle BCD) = \alpha + \beta$ and as $\alpha \neq 0$ and $\beta \neq 0$ then $\alpha + \beta > \alpha$ and $\alpha + \beta > \beta$. Thus we have in effect proved

Theorem V: *The measure of an exterior angle of a triangle is equal to the sum of the measures of the two opposite (interior) angles of the triangle and is greater than either of them.*

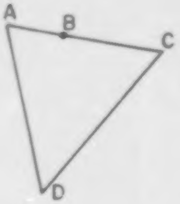
On the basis of these theorems and of the theorems of Section 1, we are now in a position to state and prove several theorems about parallels and perpendiculars. We state some of the theorems here and leave the others and all the proofs for the exercises.

Theorem VI: *If two distinct lines (in a plane) are each perpendicular to a third line then the two lines are parallel.*

Theorem VII: *If two lines are parallel, and one is perpendicular to a third line (in their plane) then the other is also.*

Theorem VIII: *Given a line ℓ and a point P. Then there is exactly one line containing P and perpendicular to ℓ.*

A quadrilateral is a simple closed curve (in a plane) which is the union of four segments (called the sides) but is not the union of three segments. (Note that a triangle is the union of four segments) and is also the union of three segments. A quadrilateral has four sides and four angles. As in the case of a triangle, we shall use the term "side" to mean either a segment or its length (as convenient).



A parallelogram is a quadrilateral in which each side is parallel to another. A parallelogram has two pairs of parallel sides.

A quadrilateral whose four angles are right angles is called a rectangle. It follows from Theorem VI that a rectangle is a parallelogram.

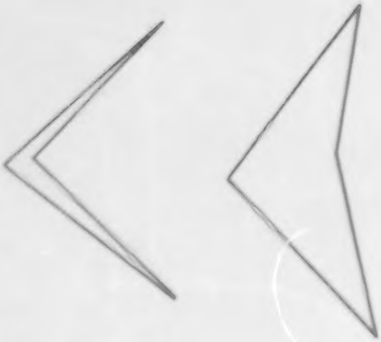
Theorem IX: The opposite angles of a parallelogram are congruent to each other.

Theorem X: The opposite sides of a parallelogram are congruent to each other (or are of equal length).

Theorem XI: The sum of the measures of the angles of a parallelogram is 360.

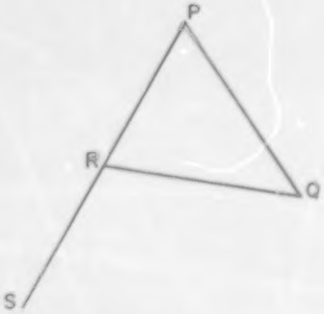
Note that from our definitions it does not follow that the sum of the measures of the angles of any quadrilateral would be 360.

Two examples are indicated on the right. The sum of certain numbers naturally associated with the angles of a quadrilateral will be 360. But these are not necessarily the measures of the angles of the quadrilateral.



Exercises 9-2

1. Write out a proof (as in the text) that m(∠QRS) = m(∠RPQ) + m(∠PQR).



2. How many exterior angles does a triangle have? How many angles are represented in a figure which is the union of 3 lines having no point in common but such that each two of them do have a point in common?
3. Prove Theorem VI. Hint: Use Theorem IV.
4. Prove Theorem VII.
5. Prove Theorem VIII. Consider two cases: One in which P is a point of ℓ, the other in which P is not a point of ℓ.

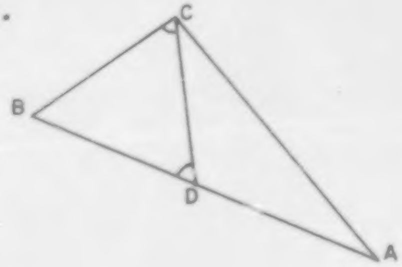
6. Prove Theorem IX.
7. Prove Theorem X.
8. Prove Theorem XI.
9. Prove that if the opposite sides of a quadrilateral are congruent to each other, then the quadrilateral is a parallelogram.
10. Show that if $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are parallel lines, the lengths of the perpendicular segments from the points of $\overleftrightarrow{AB}$ to $\overleftrightarrow{PQ}$ are all equal.

---

3. Some Inequalities Associated with Triangles--The Triangle Inequality.

In this section we list some properties without calling them theorems.

1. Consider a triangle ($\triangle ABC$).
If $AB > BC$,
then $m(\angle BCA) > m(\angle BAC)$.



Let D be a point of $\overrightarrow{BA}$ such that $\overline{BC} \cong \overline{BD}$. As $BD = BC$ and $BC < BA$, then $BD < BA$ and D is between B and A.

$\triangle BCD$ is isosceles with $\overline{BC} \cong \overline{BD}$. Hence $\angle BDC \cong \angle BCD$. Now $\angle BDC$ is an exterior angle of $\triangle CDA$ and thus $m(\angle BDC) > m(\angle BAC)$.

But because D is in the interior of $\angle BCA$, $m(\angle BCA) > m(\angle BCD)$. We have the following facts, then,

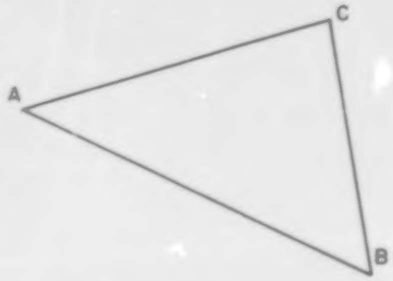
$$m(\angle BCA) > m(\angle BCD)$$

$$m(\angle BCD) = m(\angle BDC)$$

$$m(\angle BDC) > m(\angle BAC).$$

Therefore, $m(\angle BCA > m(\angle BAC)$ as was to be shown. Another way of stating this result is "If two sides of a triangle are of unequal measure, the measures of the angles opposite these sides are unequal in the same order".

2. Now we look at the converse of Statement 1. Consider a triangle $\triangle ABC$. If $m(\angle BCA) > m(\angle BAC)$, then $AB > BC$.



We prove this statement by exhausting possibilities. Either $AB > BC$ or $AB = BC$ or $AB < BC$.

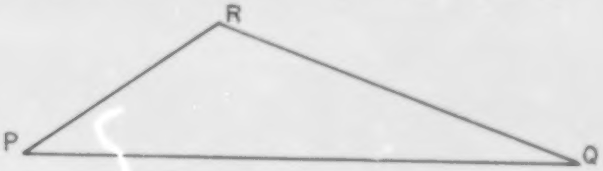
If $AB = BC$ then the triangle is isosceles and $m(\angle BCA) = m(\angle BAC)$ which is a contradiction.

If $AB < BC$ then from Paragraph 1 of this section, $m(\angle BAC) > m(\angle BCA)$ which is also a contradiction.

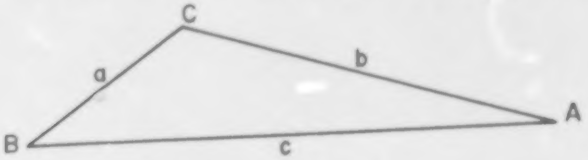
Therefore, the only possibility left is that $AB > BC$ which was to be shown.

3. We are now in a position to establish the extremely important "triangle inequality" of geometry. The triangle inequality asserts that the length of any side of a triangle is less than the sum of the lengths of the other two sides.

In one sense, the triangle inequality implies the "shortest distance" property of geometry. The straight line path from P to Q is shorter than the length of the broken-line or polygonal path from P to Q by way of R if R is not between P and Q.



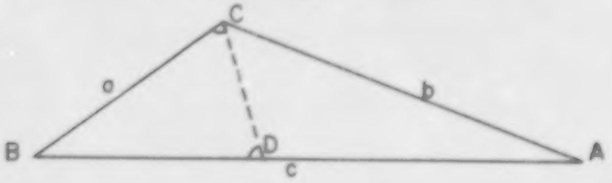
We may restate the triangle inequality as follows: If a, b, and c are the lengths of the sides of $\triangle ABC$, then $a + b > c$.



We shall agree that a, b, and c are the lengths of the sides opposite the angles at A, B, and C respectively. From a construction point of view the "triangle inequality" property is just what we expect. For if $c \geq a + b$ then in trying to construct the triangle starting with side $\overline{AB}$ the two circles with centers at B and A and radii a and b respectively would not intersect unless $c = a + b$ and then the point of intersection would be on $\overline{BA}$.

Now we give an argument based on our earlier assumptions.

We assume $c \geq a + b$.



Let D be a point on $\overrightarrow{BA}$ such that $\overline{BC} \cong \overline{BD}$. Then $\triangle BCD$ is isosceles and $m(\angle BCD) = m(\angle BDC)$. D is between B and A.

Considering $\triangle ACD$ we have that $AD \geq AC$ (even if it doesn't look like it) for $AD = c - a$ and as $a + b \leq c$ then $b \leq c - a$. But then $m(\angle ACD) \geq m(\angle CDA)$.

Now $m(\angle BDC) + m(\angle CDA) = 180$

and $m(\angle BCD) + m(\angle ACD) = m(\angle BCA) < 180$.

Hence $m(\angle CDA) = 180 - m(\angle BDC)$

and $m(\angle ACD) < 180 - m(\angle BDC)$.

Therefore $m(\angle ACD) < m(\angle CDA)$,

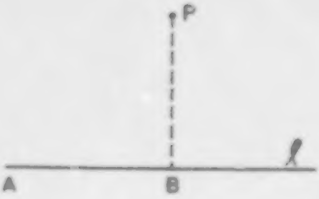
but this contradicts our earlier statement.

Hence it is not true that $c \geq a + b$.

Therefore $c < a + b$.

Exercises 9-3

1. Given three points A, B, and C. Explain how by measuring 3 distances one can find out whether or not the three points are all on the same line.



2. Suppose that ℓ is a line and P is a point not on ℓ. Suppose further that B is the foot of the perpendicular from P to ℓ and A is any other point on ℓ. Show that $PB < PA$. In other words, show that "the perpendicular distance is the shortest distance from a point to a line".

3. Let A, B, and C be the vertices of a triangle. Let P be a point which is not on the triangle but which is in the plane of the triangle. Show that the sum of the distances from P to A, B, and C is greater than $\frac{1}{2}(AB + BC + AC)$, i.e., $\frac{1}{2}$ the perimeter of the triangle.

Chapter 10

Areas, Volumes, and the Theorem of Pythagoras

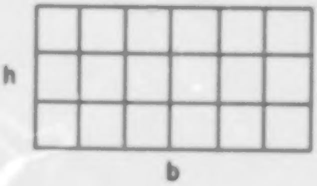
1. Areas of Parallelograms and Triangles. The Theorem of Pythagoras.

We have seen in Chapter 6 that if a rectangle has base b and height h (in terms of the same unit) then the area of the rectangle (rectangular region) is b · h (in terms of a square region of side one unit).

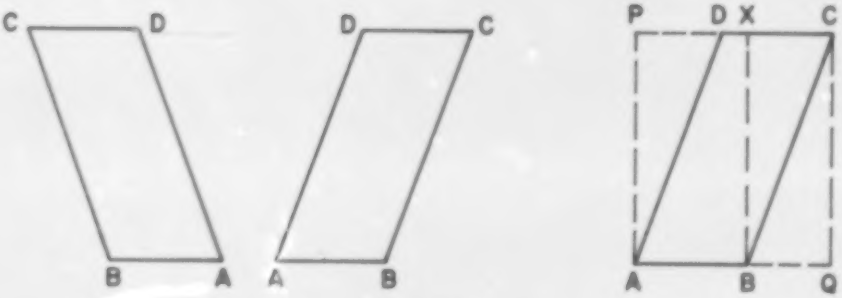
h = 3

b = 6

Area = 6 · 3 = 18



Let us develop the formula for the area of a parallelogram.



10.1

We are given the basic properties of area discussed in Chapter 6, Section 4. We will consider the parallelogram to be labeled as in the figure. Sides $\overline{AB}$ and $\overline{DC}$ can be considered to be horizontal with A and C as the "extreme" points in a horizontal sense. (Draw a figure as you read this.)

Let P, X, and Q be the feet of the perpendiculars from A and B to $\overleftrightarrow{DC}$ and from C to $\overleftrightarrow{AB}$ respectively. The point X might be D. As $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ are parallel, it follows from Theorem VII of Chapter 9, that $\overleftrightarrow{AP} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{BX} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{CQ} \perp \overleftrightarrow{CD}$. Thus $\overleftrightarrow{AP}$, $\overleftrightarrow{BX}$ and $\overleftrightarrow{CQ}$ are all parallel. Hence AQCP and BQCX are both rectangles.

Now $\triangle BXC \cong \triangle APD$ and thus from Property V of Chapter 6, Area ($\triangle$ BXC) = Area ($\triangle$ APD). From Property VI of Chapter 6, we may conclude that

Area ($\square$ BQCX) = Area ($\triangle$ BQC) + Area ($\triangle$ BXC),

and therefore Area ($\square$ BQCX) = Area ($\triangle$ BQC) + Area ($\triangle$ APD).

Again from Property VI,

Area ($\square$ AQCP) = Area (▱ABCD) + Area ($\triangle$ BQC) + Area ($\triangle$ APD).

Hence Area ($\square$ AQCP) = Area (▱ABCD) + Area ($\square$ BQCX),

or Area (▱ABCD) = Area ($\square$ AQCP) - Area ($\square$ BQCX).

From our formulas, Area ($\square$ AQCP) = (CQ)(AB + BQ)

= (CQ)(AB) + (CQ)(BQ)

and Area ($\square$ BQCX) = (CQ)(BQ).

Therefore Area (▱ABCD) = (CQ)(AB) + (CQ)(BQ) - (CQ)(BQ)

= (CQ)(AB).

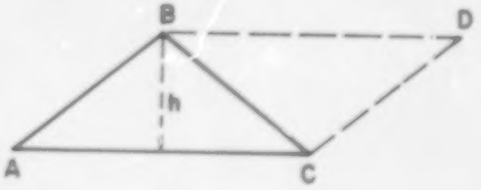
This last formula asserts that the area of the parallelogram is the product of the length of the base times the altitude. $V = b \cdot h$. This is what we wanted to show.

Note that either pair of parallel sides could have been regarded as horizontal. From Property VII of Chapter 6 we conclude that

$$V = b_1 \cdot h_1 = b_2 \cdot h_2 \qquad \text{where}$$

b_1 and b_2 are lengths of adjacent sides and h_1 and h_2 are the heights to these sides.

From the formula for the area of a parallelogram, we can very easily obtain the usual formula for the area of a triangle. Consider $\triangle ABC$. Let us regard $\overline{AC}$ as the base. Let D be the intersection of the lines through C parallel to $\overleftrightarrow{AB}$ and through B parallel to $\overleftrightarrow{AC}$.



(The assumption that the lines don't intersect means that they would be parallel which means that both $\overleftrightarrow{AB}$ and the new line through B would be parallel to the new line through C. But then we would have two lines through B parallel to a given line, for the new line through B cannot contain A and hence is different from $\overleftrightarrow{AB}$.)

Now from the SSS Theorem, $\triangle ABC \cong \triangle DCB$ and hence

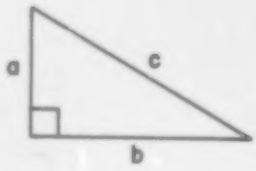
$$\text{Area}(\triangle ABC) = \frac{1}{2}\,\text{Area}(\square ACDB)$$

$$= \frac{1}{2}(AC)\cdot h = \frac{1}{2}\,b\cdot h$$

where h is the height of the triangle (and of the parallelogram).

As in the case of the parallelogram, the formula for the area of a triangle can be used with any particular side as the base.

The Pythagorean Theorem. The Theorem of Pythagoras has to do with the lengths of the sides of a right triangle. Since the sum of the measures of the angles of any triangle is 180 there can be at most one right angle in any triangle. We call the side opposite the right angle the hypotenuse of the right triangle and usually denote its length by c. The other sides are called the legs of the right triangle. We denote their lengths by a and b. The Pythagorean Theorem says that in a right triangle

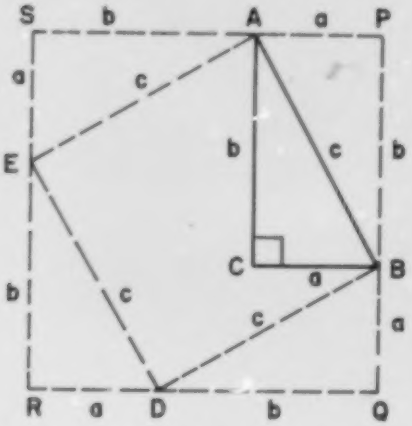


$$c^2 = a^2 + b^2 \qquad \text{or}$$

the square of the hypotenuse is equal to the sum of the squares of the other two sides.

There are a tremendous number of "proofs" of the Pythagorean Theorem. Even President Garfield once gave such a proof. We give one of the more elementary geometric ones.

In this proof we assume some properties of rectangles which we have not stated explicitly, but which follow from observations we have made in the previous chapter.



In this paragraph we describe the figure above. We are given the right triangle $\triangle ABC$. There exist lines through A and B perpendicular to $\overleftrightarrow{AC}$ and $\overleftrightarrow{CB}$ respectively. Let P be the point of intersection of these lines. APBC is a parallelogram (rectangle) and hence AP = BC = a while PB = AC = **b**. Let Q and S be points on $\overleftrightarrow{PB}$ and $\overleftrightarrow{PA}$ respectively as in the figure such that BQ = a and AS = b. There exist lines perpendicular to AS at S and BQ at Q respectively. Let R be their point of intersection. PQRS is a rectangle with adjacent sides equal in length. Hence PQRS is a square. Let E be a point of $\overline{SR}$ and D a point of $\overline{RQ}$ such that SE = RD = a. Then it may be observed that ABDE is a square of side c. We leave the proof to the exercises.

$$\text{Area } (\square\, PQRS) = (a + b)(a + b) = a^2 + 2ab + b^2$$

$$\text{Area } (\square\, ABDE) = c^2.$$

$$\text{Area } (\triangle\, APB) = \text{Area } (\triangle\, BQD) = \text{Area } (\triangle\, DER) = \text{Area } (\triangle\, AES)$$
$$= \frac{1}{2}\, a \cdot b$$

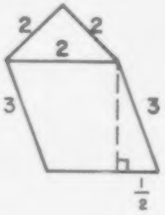
$$\text{Area } (\square\, PQRS) = \text{Area } (\square\, ABDE) + \text{Area } (\triangle\, APB) + \text{Area } (\triangle\, BQD)$$
$$+ \text{Area } (\triangle\, DER) + \text{Area } \triangle\, (AES)$$

Therefore $a^2 + 2ab + b^2 = c^2 + 4\cdot(\frac{1}{2}ab)$

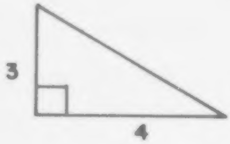
$$a^2 + b^2 + 2ab = c^2 + 2ab$$

Hence $a^2 + b^2 = c^2$ as was to be shown.

Exercises 10-1



1. Find the area of the region of the figure on the right.



2. Find the altitude to the hypotenuse of the right triangle of the figure.
   (Hint: Equate two expressions for the area.)

Exercises 3 and 4 refer to the description of the figure in the discussion about the Pythagorean Theorem.

3. Prove that the lines through A and B perpendicular to $\overleftrightarrow{AC}$ and $\overleftrightarrow{CD}$ respectively must intersect.

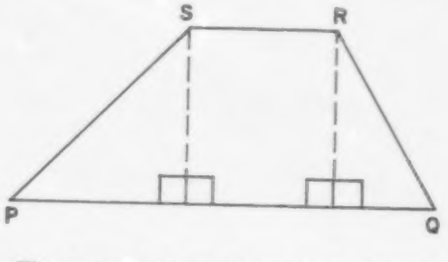
4. Prove that ABDE is a square. Note that the sides are of equal length (from the congruent corner triangles). Hence show one of the angles is a right angle.

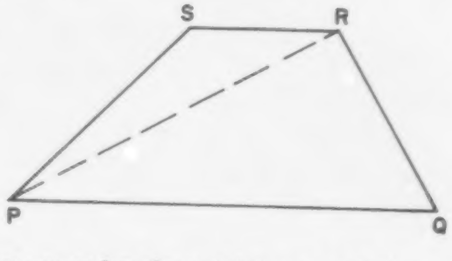
---

2. Other Areas and Decompositions.

There are various other figures for which we want to compute areas. Some of these are more complicated closed regions in the plane and some are surfaces or parts of surfaces of solids. In general, the approach to computing the areas of such figures is to think of the figures as the union of simple figures. Then we may compute the areas of the various simple figures. In some cases we develop special formulas and use them for computations. But in many instances, it is easier to remember the geometric considerations which lead to the formulas than to remember the formulas as such. (An exception is the formula for the area of a parallelogram, which can be considered to be like that for a rectangle.)

A trapezoid is a quadrilateral with two parallel sides such that the other two sides are not parallel.

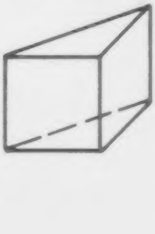


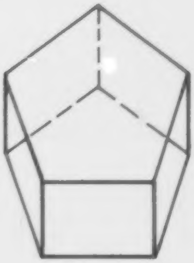


The area of a closed trapezoidal region may be found by one of two standard devices. We may decompose it into two right triangular and one rectangular region as on the left or into two triangular regions as on the right. From either of these we can derive the usual formula for the area as $\frac{1}{2} h(b_1 + b_2)$ where h is the altitude (perpendicular distance between parallel sides) and b_1 and b_2 are the lengths of the bases.

In applications of geometry, there are a number of problems which arise as to the total surface area of a prism or pyramid or the lateral surface area of such. The distinction between "total surface" area and "lateral surface" area is the following: If the solid concerned has bases (one or two) then the lateral surface area refers to the area of the union of the faces other than the base(s) whereas the total surface area refers to the area of the union of all faces.

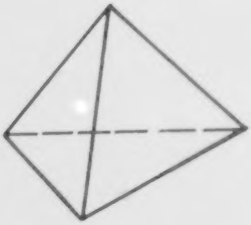
Among solids that are commonly dealt with are prisms and pyramids. A prism is a polyhedron (a solid with flat faces) such that some two faces are congruent and are in parallel planes. These faces are called the bases. The other faces are all parallelograms (or rectangles for right prisms) and each of these parallelograms has a pair of opposite edges in the two bases. A triangular prism is a prism whose bases are triangles. The figure on the right represents a triangular prism. The one below it is a prism with pentagons for bases.

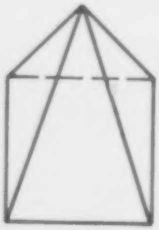




A pyramid is a polyhedron with one face designated as a base and with all the other faces being triangular, having a vertex in common and having the other two vertices of each on the base.

The figures below represent triangular and square pyramids (the adjectives describing the bases).





A much more comprehensive treatment of polyhedrons is given in Chapter 14. The "solid polyhedrons" of this chapter are really 3-dimensional polyhedrons.

Exercises 10-2

1. Derive the formula for the area of a trapezoidal region by decomposing the region into two triangular regions.
2. Derive the formula for the area of a trapezoidal region by decomposing the region into 2 right triangular and one rectangular region.
3. (a) Find the lateral surface area of a right prism in terms of the perimeter of the base and the height of the prism.
   (b) Find the total surface area in terms of the result of (a) and the areas of the bases.

4. Suppose a square pyramid of side 8 has its triangular faces all congruent to each other. Suppose the slant height (altitude of one of the triangular faces) is 10.
   (a) Find the lateral surface area.
   (b) Find the total surface area.

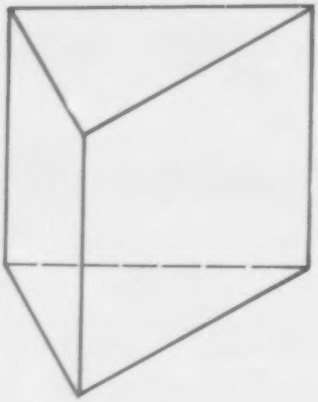
---

3. Volumes.

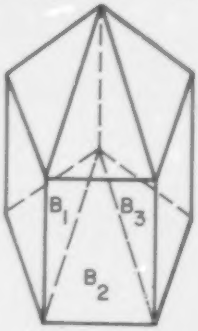
We seek a point of view which enables us to find the volume of a complicated polyhedral region (i.e. interior of a polyhedron together with its boundary). As before we think of decomposing the solid region into simpler ones--i.e. we think of expressing the complicated solid region as the union of non-overlapping simpler regions. (As noted in the last section, Chapter 14 has a much more comprehensive treatment of polyhedrons.)

For volumes, we have available, so far, the volume of a rectangular parallelepiped. It is either $b \cdot d \cdot h$ (base x depth x height) or $B \cdot h$ where $B = b \cdot d$ and is the area of a rectangular region which is regarded as the base.

The point of view $V = B \cdot h$ turns out to be a useful one. It is also applicable to prisms (and to cylinders as discussed in Chapter 11).



Let us start with a triangular right prism. By a construction and argument like that given for parallelograms and triangles in Section 1 of this chapter we can decide that the volume of the triangular prism is $\frac{1}{2}$ that of a rectangular parallelepiped whose bases are rectangles of area twice that of the triangular bases. Thus it follows that the volume of the triangular prism is $B \cdot h$. Now, any right prism can be decomposed into non-overlapping triangular right prisms. We simply have to decompose the base region into triangular regions. Then the volume of the prism is the sum of the volumes of the triangular prisms. For our figure



$$V = B_1 \cdot h + B_2 \cdot h + B_3 \cdot h$$

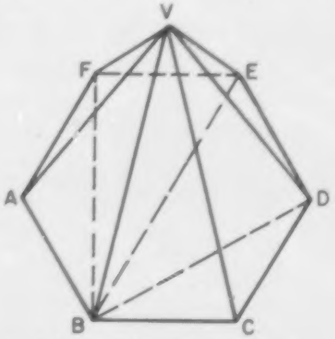
$$= (B_1 + B_2 + B_3) \cdot h$$

$$= B \cdot h$$

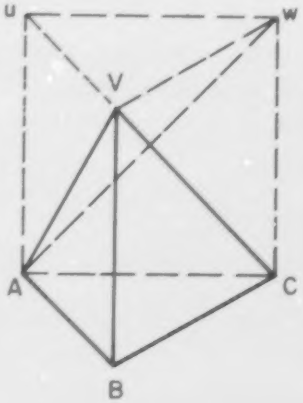
where B_1, B_2, and B_3 are the areas of the three base triangles and B is the area of the pentagon. The formula $V = B \cdot h$ is also applicable to oblique prisms (prisms that are not right prisms). The height h is the perpendicular distance between the planes which contain the bases.

Similarly the volume of any pyramid can be expressed as the sum of the volumes of triangular pyramids by decomposing the base into triangular regions and using the vertex of the pyramid as the vertex of all of the triangular pyramids.

The hexagonal pyramid of our drawing is expressed as the union of four triangular pyramids.



We now seek the volume of a triangular pyramid. The volume is $\frac{1}{3}(B \cdot h)$. A "proof" of the formula uses what is known as Cavalieri's Theorem and more mathematical apparatus than we choose to use here. Rather we shall simply try to make it seem reasonable. Consider lines through V parallel to $\overleftrightarrow{BC}$ and $\overleftrightarrow{BA}$ respectively. Using $V \longleftrightarrow B$, and points U and W on these lines, there exists $\triangle$ UVW which is congruent to $\triangle$ ABC and is in a plane parallel to the plane of $\triangle$ ABC.



Now prism (ABC) - (UVW) can be decomposed into 3 triangular pyramids V - ABC, A - UVW and C - AVW. The base of the last of these is not a base of the prism. It seems reasonable and can be proved by use of Cavalieri's Theorem that all three have the same volume. Therefore pyramid V - ABC has $\frac{1}{3}$ the volume of the prism but the prism has the same base - ABC - and the same height as the pyramid. Hence $V_{(pyramid)} = \frac{1}{3}(B \cdot h)$.

Returning to the case of a general pyramid, we note that the altitudes of the triangular pyramids we get are all the same as the altitude of the original when we consider them all to have bases in the plane of the original base.

Thus $V = \frac{1}{3}(B_1 \cdot h) + \frac{1}{3}(B_2 \cdot h) + \dots + \frac{1}{3}(B_k \cdot h)$

$= \frac{1}{3}(B_1 + B_2 + \dots + B_k) \cdot h = \frac{1}{3}(B \cdot h)$.

The formula $V = \frac{1}{3}(B \cdot h)$ is the formula that we were seeking.

Exercises 10-3

1. Find the volume of an oblique prism whose base is a (2 by 5) rectangle and whose perpendicular distance between faces is 12.
2. Find the volume of a pyramid whose altitude is 8 and whose base is a regular hexagonal region of side 2". A hexagon is regular if all its sides are congruent and all its angles are congruent.

3. Find the volume of a right prism whose height is 10 and whose base is pentagonal as in the figure.

4. Find the volume of a pyramid whose height is 6, whose base is a parallelogram as in the figure.

5. Draw figures illustrating problems (1) through (4).

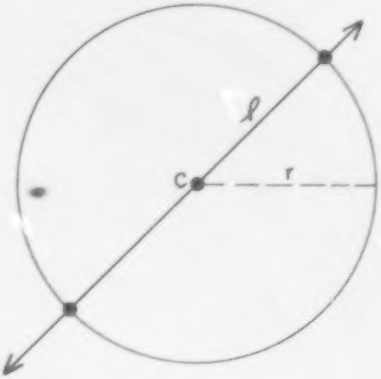
Chapter 11

Circles, Cylinders and Cones

1. Terminology.

For the first part of this chapter we deal with sets in the plane. In the final part we shall deal with cylinders and cones in space.

Let C be a point and let r be a number. Then the circle with center at C and radius r is the set of all points (of the plane) at a distance r from C. Let ℓ be any line which contains C. On ℓ there are two rays with endpoint at C. On each of these there is exactly one point of the circle, for on each there is exactly one point at distance r from C. Any line through C, therefore, contains exactly two points of the circle.

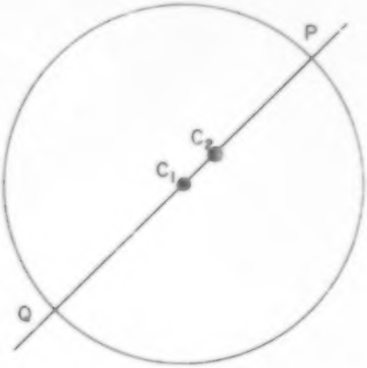


We usually draw a (representation of a) circle by use of a compass. We draw the circle in such a way that it fits our description of a simple closed curve. We start drawing and without lifting the pencil draw until we return to the point we started with. Except for the first point we cover each point only once. Thus a circle is an example of a simple closed curve. We can, in the case of a circle, say exactly what we mean by its interior and by its exterior.

The interior of the circle with center C and radius r is the set of all points at a distance less than r from C. The exterior is the set of all points at a distance greater than r from C.

A circle is a curve. It is not the curve together with its interior. A circle has a center (exactly one center, in fact) and a radius. The center is a point but the radius (as we have used it) is a number (or length in some contexts). Sometimes the term radius is also used to denote a segment having one endpoint at the center of the circle and having the other endpoint on the circle. In traditional terminology the term "radius" is used in both these senses. Little confusion results from this as it is usually clear which sense is meant. We, too, shall use the term radius with both meanings.

Let us now prove that a circle cannot have two centers. Suppose C_1 and C_2 were distinct points and were both centers of the circle. The line $\overleftrightarrow{C_1C_2}$ must intersect the circle in two points. Call them P and Q as in the figure with C_1 between Q and C_2 as in the figure.



Now $QC_1 = C_1P$ as C_1 is a center of the circle, and $QC_2 = C_2P$ as C_2 is a center of the circle.

Also $QC_1 < QC_2$.

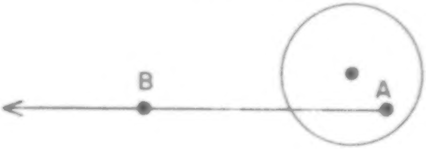
Thus we have $C_1P = QC_1$

$$QC_1 < QC_2$$

$$QC_2 = C_2P$$

Therefore, $C_1P < C_2P$ but, from the order of the points on the line, $C_1P > C_2P$. We have a contradiction. Hence a circle can have at most one center.

Let us consider another basic property of circles. Let D_1 be a circle. If A is in the interior of D_1 and B is in the exterior then $\overrightarrow{AB} \cap D_1$ is exactly one point. We do not prove this property. However, let us note that it agrees with our earlier observations about the interior and exterior of any simple closed curve. The segment $\overline{AB}$ is a polygonal path from A to B and hence must intersect the simple closed curve.

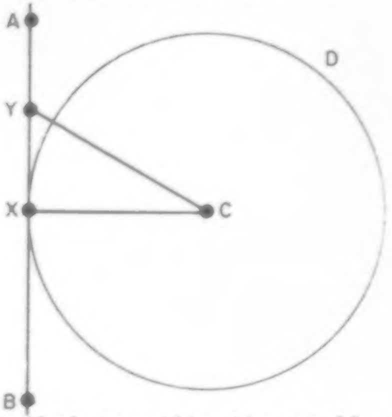


A tangent to a circle is a line that intersects the circle in exactly one point. It follows from our observation above about rays that a tangent to a circle cannot contain a point of the interior of the circle.

We state one of the standard fundamental properties about circles and tangents.

Property 1: *If D is a circle with center C and $\overleftrightarrow{AB}$ is tangent to D at point X, then $\overleftrightarrow{AB}$ is perpendicular to $\overleftrightarrow{XC}$.*

Proof: There is a line through C perpendicular to $\overleftrightarrow{AB}$. Let Y be the intersection of $\overleftrightarrow{AB}$ and this line. Suppose Y is not X. Now $\angle CYX$ is a right angle. The sum of the measures of the angles of ΔCXY is equal to 180. Therefore $m(\angle CXY) < 90$. Hence $m(\angle CXY) < m(\angle CYX)$. The side opposite the larger angle is longer than that opposite the smaller angle. Therefore $CX > CY$ and hence Y must be in the interior of D for CX is the radius. But, as we have observed, a tangent to a circle cannot contain a point in the interior of the circle. Therefore our assumption that Y is not X is false. Y must be X, $\overleftrightarrow{CY}$ is $\overleftrightarrow{CX}$, and thus $\overleftrightarrow{CX}$ is perpendicular to $\overline{AB}$.



Exercises 11-1

1. Prove that a line cannot intersect a circle in a set consisting of three or more points.

2. A chord of a circle is a segment whose endpoints are points of the circle. Consider a chord that does not contain the center of the circle. Prove that the line containing the midpoint of the chord and the center of the circle is perpendicular to the chord.
3. Prove that if D and E are distinct circles then $D \cap E$ cannot be a set consisting of three or more points.

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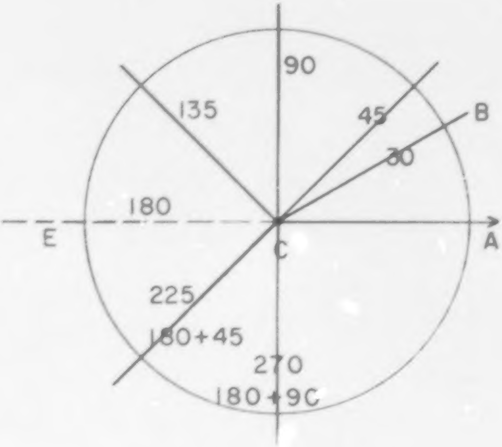
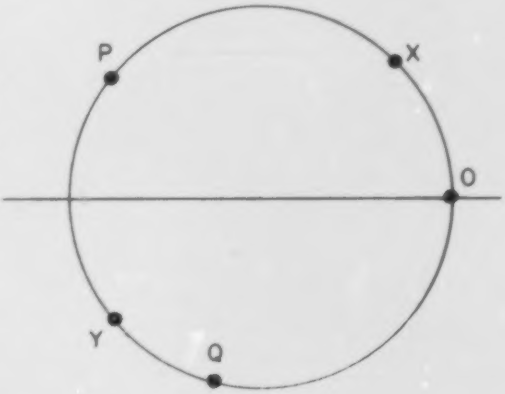
2. Arc Measure and Length.



Consider a circle with center C. Let $\overrightarrow{CA}$ be a ray. For convenience we think of $\overleftrightarrow{CA}$ as horizontal with A to the right of C. Let E be a point of $\overleftrightarrow{CA}$ not on $\overrightarrow{CA}$. Let B be a point not on $\overleftrightarrow{CA}$. For convenience let us take B above the line $\overleftrightarrow{CA}$. Now in Chapter 6 we saw that the family of all rays with endpoint at C and containing points on the B-side of $\overleftrightarrow{CA}$ could be coordinatized using numbers from 0 to 180. We called our unit a degree. For those rays which contain points on the non-B-side of $\overleftrightarrow{CA}$ we choose to coordinatize them by adding 180 to their degree coordinates which one would get by starting with $\overrightarrow{CE}$ as the reference (or zero) ray. The ray $\overrightarrow{CA}$ is considered as having two alternative coordinates--0 or 360. The ray $\overrightarrow{CE}$ is the 180 (degree) ray.

This coordinatization of the family of all rays with endpoint at C induces a coordinatization of the set of points of the circle. Each point of the circle is identified with the coordinate of the ray containing it.

Suppose P and Q are any two points of a circle. The set $\{P, Q\}$ separates the circle into two sets. The union of either of these and $\{P, Q\}$ is called an arc of the circle. The symbol $\overset{\frown}{PXQ}$ is used to denote the arc which contains X and has endpoints P and Q. Note that $\overset{\frown}{PXQ} \cup \overset{\frown}{PYQ}$ is the circle of the figure above.



We can now define what we mean by the degree measure of an arc. We may consider the circle to be coordinatized as above.

Case I: If $\overset{\frown}{PYQ}$ does not contain the point with zero coordinate then the degree measure of $\overset{\frown}{PYQ}$ is the positive difference in the coordinates of P and Q.

Case II: If $\overset{\frown}{PYQ}$ *does* contain the point with zero coordinate and neither P nor Q is such point then the degree measure of $\overset{\frown}{PYQ}$ is 360 minus the positive difference in the coordinates of P and Q.

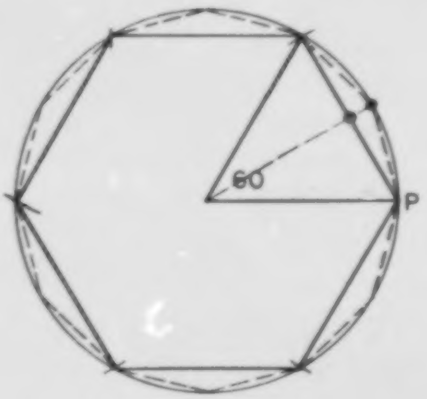
Case III: If P or Q is the point with zero coordinate and the arc $\overset{\frown}{PYQ}$ does not contain other points with coordinates close to 360 then the degree measure of $\overset{\frown}{PYQ}$ is the positive difference in the coordinates of P and Q with zero as the coordinate of P or Q.

Case IV: If P or Q is the point with zero coordinate and the arc $\overset{\frown}{PYQ}$ does not contain other points with coordinates close to zero then the degree measure of $\overset{\frown}{PYQ}$ is the positive difference in the coordinates of P and Q with 360 as the coordinate of P or Q.

The degree measure of an arc is *not* the "length" of the arc. Rather it is the measure of the amount of "turning" of the arc. The closer the arc is to a whole circle, the closer the degree measure is to 360.

An arc of a circle with degree measure less than 180 determines an angle whose vertex is the center of the circle and whose rays contain the endpoints of the arc. We call such angle a central angle. The measure of the central angle is the degree measure of the arc determining it. For some purposes, it is convenient to think of any arc of a circle as determining a "central angle" whose measure is the degree measure of the arc. This allows "central angles" to have degree measures anywhere from 0 to 360.

Length. Intuitively we know that a circle must have length. We can wrap a string around a circular object and then measure it. We can mark a point on a bicycle wheel tire at contact with the ground and note the length of the path made by rolling the wheel until the marked point returns to contact with the ground. Experimentally, the answer comes out to be somewhat more than 6 times the radius (i.e., $2\pi r$). Sometimes the length of a circle is called its circumference.



Now, mathematically, if we want to measure the length of a circle we can think about doing it in the following way. Starting from a point P on the circle lay off the radius in straight line segments six times. Then the central angle subtended (determined) by each chord is a 60 degree angle for we have equilateral triangles formed. Hence we would have inscribed a hexagon in the circle. The hexagon is called regular in that all of its sides are congruent and all of its angles are congruent. It seems clear then that the length of the circle is greater than six times the radius. But the number 6r can be considered as an approximation to the length of the circle. Now we can bisect each of the 6 central angles (by finding the midpoints of the chords if we wish) and determine 6 more points on the circle. Using the original 6 and the 6 additional ones we could construct a regular 12-sided polygon. Its length (perimeter) could be computed (or measured) and we should have a better approximation for the length of the circle. The process can be continued to produce a regular 24-sided polygon, then a regular 48-sided one, etc. At each stage the length of the polygon is less than that of the circle but close to it. The length of the circle is the least number which exceeds the lengths of all the inscribed polygons so obtained. The ratio of this least number to 2r (twice the radius) is called π. Thus the length of the circle is $2\pi r$. It can be established that $\pi = 3.141592 \ldots\ldots$ As we have been led to expect, 2π is somewhat more than 6.

The number, π, exists in the nature of things. Nobody has any control over its value. We can think of π as being bracketed between successive whole numbers, then tenths, then hundredths, etc. Thus

$$3 < \pi < 4$$

$$3.1 < \pi < 3.2$$

$$3.14 < \pi < 3.15$$

$$3.141 < \pi < 3.142$$

etc.

It turns out that the decimal expansion for π is not a repeating decimal expansion, i.e., π is not a rational number. Sometimes the rational number $\frac{22}{7}$ is used as an approximation for π. However, $\frac{22}{7}$ is not π, it is simply close to π. We might write $\pi \approx \frac{22}{7}$. Computations of π to over 10,000 decimal places have been made in recent years.

If a circle has length, then arcs of the circle should also have length. The degree measure of an arc is a certain number between 0 and 360. In a sense, 360 is the degree measure of a circle. Because congruent arcs should have equal lengths and because two arcs of the same degree measure and on the same circle are congruent, we can say that

$$\frac{\text{length (arc)}}{\text{length (circle)}} = \frac{\text{degree measure (arc)}}{\text{degree measure (circle)}}.$$

In other words

$$\text{length (arc)} = \frac{\text{degree measure (arc)}}{360} \cdot 2\pi r.$$

Thus, for example, the length of a semi-circle is

$$\frac{180}{360} \cdot 2\pi r \text{ or } \pi r \text{ as our}$$

intuition tells us it ought to be.

Important questions come up with respect to how to use π in computations. The question, "What is the length of a circle of radius 10?" has an answer which can be written in the form 20π. Clearly 20π is a perfectly good number. It is the product 20 times π. Numerically it is between 62 and 63. A decimal approximation of 20π accurate to 2 decimal places is 62.83. We have already learned in Chapter 7 that in practical problems, if the radius of a circle is given as 10, then our convention calls for an assumption of precision either to the nearest 10 units or to the nearest unit. Thus in a practical problem, any answer for the length of the circle which carries more than two significant digits is really essentially unjustified. We should write the answer as 63 or leave it in the form 20π.

In the formula, circumference $= 2\pi r$ the number 2 is regarded as completely accurate, π as completely accurate, and r as being as accurate as we choose to give it. The number of significant digits we use for π should *not* appreciably exceed the number of digits to which r is assumed accurate.

If there is a desire to get students to use several decimal places of π for computational practice, then specific instructions to this effect can be given. *But in a practical problem accuracy of an answer should not be stated or implied beyond that justified by the measurements concerned.* To do the contrary is to give a wrong answer, an answer which is definitely deceiving, an answer which asserts precision which is simply not there.

Exercises 11-2

1. Write out the first four places of the decimal expansion of $\frac{22}{7}$. Compare with the value of π given in the text. Thus, show that $\frac{22}{7} \neq \pi$.
2. Draw two arcs whose degree measures are each 60 but such that one is twice the length of the other. What can you say about the radii of the circles which contain these arcs?
3. Using the result of Exercise 3 of Section 1, explain why an arc can be a subset of only one circle. In other words, if an arc is determined, *the* circle which contains the arc is determined.
4. Give examples and draw figures illustrating Cases I, II, III, and IV for an arc of degree measure 60.

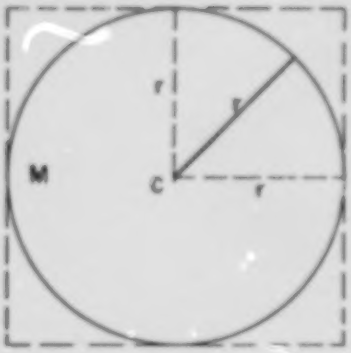
5. Find the length of an arc of degree measure 120 if the circle containing the arc has radius 8.

6. In finding the circumference of a circle whose radius is measured as indicated, what approximation should you use for π and to how many significant digits should you express the answer? (There may be questions of judgment in some cases.)
   (a) $r = 8.$
   (b) $r = 8.0.$
   (c) $r = 8.02.$
   (d) $r = 8.021.$
   (e) $r = 8.0214.$

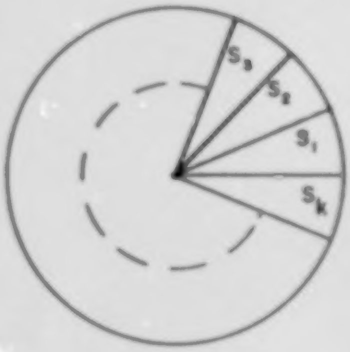
7. From the formula length = $2\pi r$ it is possible to find either the length or the radius if the other is known. Also as the "diameter" d is twice the radius, knowledge of the diameter or radius yields knowledge of the other. Find the other two of ℓ, r, and d if
   (a) $\ell = 20.$
   (b) $r = 4.$
   (c) $d = 12.$

3. Area of a Circular Region.

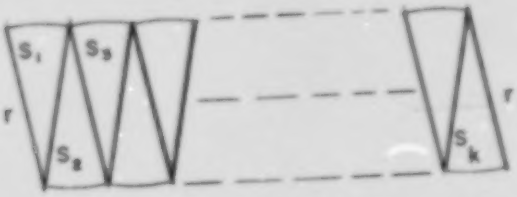
Consider a circle with center C and radius r. The circle is a simple closed curve. Let M be the closed region bounded by the circle. In Chapter 6 we have stated that with respect to a given unit (square) region there is a number which represents the area of M. For simplicity, we sometimes talk about the area of a circle and mean the area of the closed region bounded by the circle.



Our problem is to get an expression or formula for the area of M. We might note as a first approximation, that the area is clearly less than $4r^2$, for M is contained in a square region of area $4r^2$. We would guess, probably, that the area would be related to the number π as introduced in the previous section. To develop the formula for the area of M we use something of a trick. We think of expressing M as the union of non-overlapping sectors all congruent to each other. Let us suppose that we have k of them and that k is an even number. We call them S_1, S_2, —, S_k.



By Properties V and VI of Chapter 6 the area of each S_1 is $\frac{1}{k} \cdot$ (Area M). Now the area of the circular region M is clearly the area of the region represented below.



This region is bounded by a simple closed curve. It is somewhat like a rectangle or a parallelogram. However, the top and bottom are not segments but unions of arcs of circles. If k is a large even number then the region is very much like a rectangular region. The area of the region which is almost rectangular should be approximately the height times the length of the base. For large k, the height is almost r and the length of the "base" is 1/2 the length of the circle. Therefore the area should be approximately as indicated below:

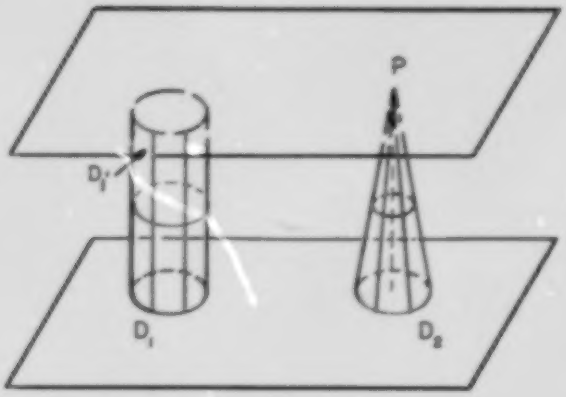
$$\text{Area} \approx r \cdot \frac{1}{2}(2\pi r) = \pi r^2.$$

For very large k, the formula is very close to being correct as the figure is almost a rectangular region. Hence we seem justified in concluding that $\text{Area}(M) = \pi r^2$ since the area of M is the area of each of these odd shaped regions we have been considering.

11.16

Areas and Volumes of Cylinders and Cones. A cylinder, a cone, and a sphere are geometric objects in space whose descriptions either depend on or are like that of a circle. We shall investigate the sphere in Chapter 13. Here we consider the cylinder and the cone and we restrict ourselves to right circular cylinders and right circular cones. The definitions given here are for application to mensuration formulas. Somewhat different definitions may be given in other contexts.

Consider two parallel planes which we shall regard as horizontal. Let D_1 and D_2 be circles in the lower plane as in the figure. Let D_1' be a circle in the upper plane with D_1' directly above D_1 (and congruent to it). Let P be a point in the upper plane directly above the center of D_2 .



The cylinder with bases D_1 and D_1' (or more precisely the closed regions bounded by D_1 and D_1') is the union of all vertical segments each of which has one endpoint in D_1 and the other in D_1'.

The cone with base D_2 and vertex P is the union of all segments each of which has one endpoint P and the other in D_2.

From some points of view it is convenient to regard the "cylinder" and the "cone" as containing the circular regions which are bases of these sets. With the bases included, then the regions bounded by the "cone" and the "cylinder" have volume.

A cylinder and a cone each has area called its lateral surface area. The sum of this area and the area of its bases (or base) is called the total surface area of the cylinder (or cone). There are very close analogies between a "cylinder" and a prism and between a "cone" and a pyramid. In fact, the cylinder and the cone can be regarded as "limiting cases" of a prism and a pyramid respectively by regarding the base circles as "limiting cases" of regular polygons as in Section 2. Thus it is reasonable to conclude that the formulas for volume, lateral surface area, and total surface area are like those for prisms and pyramids. We consider h the distance between the base planes, r the radius of the base circle, and ℓ the length of a segment from P to D_2.

$$\text{Volume (cylinder)} = h(\pi r^2) = \pi r^2 h$$
$$\text{Area (lateral surface of cylinder)} = h(2\pi r) = 2\pi rh$$
$$\text{Area (total surface of cylinder)} = 2\pi rh + 2\pi r^2$$

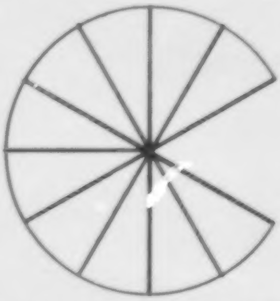
$$\text{Volume (cone)} = \frac{1}{3}h(\pi r^2) = \frac{1}{3}\pi r^2 h$$
$$\text{Area (lateral surface of cone)} = \frac{1}{2}(2\pi r)\ell = \pi r\ell$$
$$\text{Area (total surface of cone)} = \pi r\ell + \pi r^2 .$$

It is not important to remember these formulas as such. It is important to be able to think of the geometry of the situation and thus to recognize what the formulas must be.

Exercises 11-3

1. Explain why the figure of the first part of this section would be like a trapezoid if k were odd.
2. In terms of the properties of Section 4 of Chapter 6, explain why the area of M is the area of the odd-shaped figure used.
3. The label on an ordinary tin can represents a cylinder (the way we have defined it). The label may be laid flat and forms a rectangular region. The area of the label is the lateral surface area of the cylinder. Explain the formula from this point of view.

4. An ordinary conical drinking cup represents a cone (the way we have defined it). If the cup is slit to the vertex, the paper may be laid flat forming a circular region with a sector removed. The area of the paper is the lateral surface of the cone. Explain the formula for lateral surface area from this point of view.



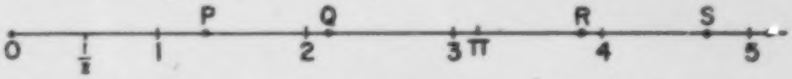
5. Compare the geometric points-of-view for area and volume of (a) a prism, and (b) a cylinder.
6. Compare the geometric points-of-view for area and volume of (a) a pyramid, and (b) a cone.
7. Find volume, lateral surface area and total surface area of a "cylinder" of height 8" and circumference of the base 18π.
8. Find volume, lateral surface area and total surface area of a "cone" of height 8" and radius of the base 6". (The "slant height" ℓ can be found by use of the Pythagorean Theorem.)

Chapter 12

The Coordinate Plane and Graphs

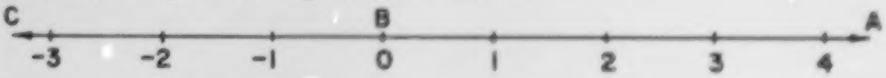
1. The Coordinate Line and the Coordinate Plane.

In Chapter 6 we have observed that a ray may be coordinatized with any segment as a unit. This coordinatization of the ray gives a one-to-one correspondence between the set of positive real numbers and zero and the set of points of the ray. We correspond zero to the end point of the ray.



The correspondence preserves order in the following sense. If P, Q and R are any three points of the ray with Q between P and R then the number corresponding to Q is between the numbers corresponding to P and R. The correspondence also preserves distance in the following sense. If $\overline{PQ} \cong \overline{RS}$, then, of course, PQ = RS and further, PQ (the length of $\overline{PQ}$)is the absolute value of the difference between the coordinates of P and Q. In the figure, PQ is approximately .9. A similar statement is true about RS. To coordinatize the whole line we coordinatize a ray $\overrightarrow{BA}$ of the line with $B \longleftrightarrow 0$. Let C denote a point of $\overrightarrow{AB}$ but not of $\overrightarrow{BA}$. Then we coordinatize $\overrightarrow{BC}$ with the same unit segment.

Now if we think of assigning negative values to the points of $\overrightarrow{BC}$ instead of the corresponding positive values we have the usual coordinatization of the line. In this coordinatization order and "distance" are preserved as in the case of the ray. We customarily think of coordinatizing a horizontal line with the points with positive coordinates being to the right of the zero point. Now by thinking of the line we

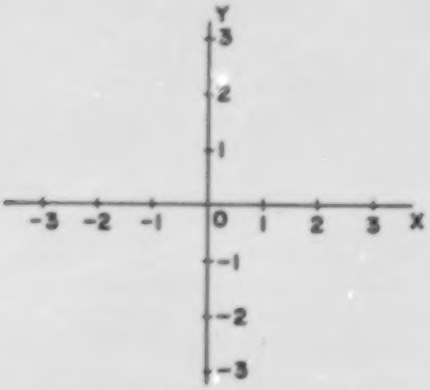


can easily tell what we mean by the statement $a < b$. We say a is less than b or $(a < b)$ if the point whose coordinate is a is to the left of the point whose coordinate is b. For example,

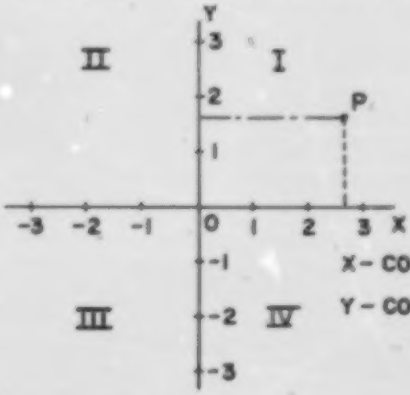
$$-2 < 16$$
$$-2 < 1$$
$$-2 < 0$$
$$-2 < -1$$
$$-5 < -2$$

We also say that b is greater than a or $(b > a)$ if the point corresponding to b is to the right of the point corresponding to a. We use the symbol "$\geq$" to mean "greater than or equal to." Note that $c \geq d$ means geometrically that the point whose coordinate is c is not to the left of the point whose coordinate is d.

Having in mind the principles of coordinatization of the line we can now easily coordinatize the plane. Think of two perpendicular lines. Consider one as horizontal. We call the point of intersection of the two lines the origin and label it by 0 (oh).



Coordinatize the horizontal line with positive coordinates to the right and the vertical line with positive coordinates upward. We customarily use the same unit for both lines. We call the two coordinate lines the *axes*, calling the horizontal one the x-*axis* and the vertical one the y-*axis*. We may label the axes with our scale and put the letters x and y as indicated to the right and up. Now to coordinatize the plane we think of ordered (or sensed) pairs of numbers. The "ordered" means that in general (a,b) is not the same as (b,a). Each ordered pair (a,b) is to correspond to one point of the plane and each point to one ordered pair of numbers. We set up the one-to-one correspondence as follows.



X-COORDINATE OF P IS $2\frac{2}{3}$

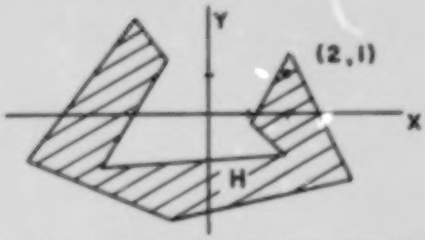
Y-COORDINATE OF P IS $1\frac{3}{5}$

For point P of the plane, the coordinate on the x-axis of the foot of the perpendicular from P to the x-axis is called the x-coordinate of P. Similarly the coordinate on the y-axis of the foot of the perpendicular from P to the y-axis is called the y-coordinate of the point P. We write the x-coordinate as the first number of the ordered pair, the y-coordinate as the second. Note that the y-coordinate of any point on the x-axis is zero. What is the x-coordinate of any point on the y-axis? The coordinatization process we have described clearly gives us a one-to-one correspondence of the type we seek. Given the axes, for any point there is a unique ordered pair of real numbers, and for any ordered pair of real numbers there is a unique point. In the exercises, we develop this aspect further.

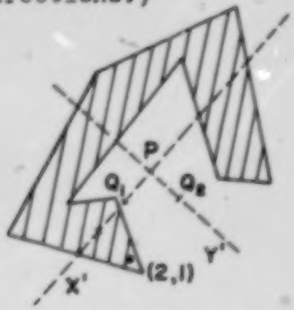
The union of the axes separates the plane into 4 sets of points. Any one of these, together with its boundary, is called a quadrant. We designate the upper right hand quadrant as the first quadrant, the upper left as the second, the lower left as the third and the lower right as the fourth.

Having the concept of a coordinate plane we now can state exactly what is meant by saying that any figure in the plane can be freely moved without changing its size or shape.

Let H be a certain set of points. In the figure H is the closed region bounded by the simple closed curve.



Suppose we are given any point P and any two points Q_1 and Q_2 such that $PQ_1 = PQ_2 = 1$ and such that $\overleftrightarrow{PQ_1}$ is perpendicular to $\overleftrightarrow{PQ_2}$. Then coordinate axes exist with P as the origin, Q_1 the point (1,0) and Q_2 the point (0,1). (We do not have further control over positive directions.)



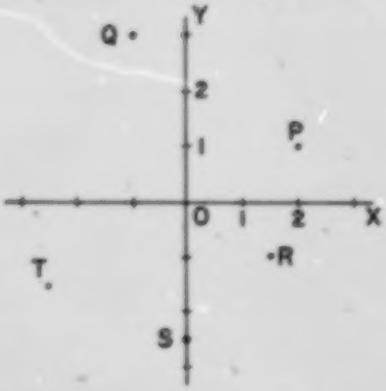
We label the axes as the x' and y' axes (the x-prime and y-prime axes).

Now let H' be the set of all points whose coordinates with respect to the x' and y' axes are the coordinates of a point H with respect to the x and y coordinate axes. For example, the point (2,1) (with respect to x and y) is a point of H. The point (2,1) (with respect to x' and y') is required to be a point of H'. It will be true that H' is congruent to H. We have "freely moved" H to H' because we have been able to choose the point P and the points Q_1 and Q_2 freely subject only to the restriction that $PQ_1 = PQ_2 = 1$ and $\overleftrightarrow{PQ_1}$ is perpendicular to $\overleftrightarrow{PQ_2}$. Note that because $PQ_1 = PQ_2 = 1$, we are saying that the unit distance in the x'y'-plane is the same as the unit distance in the xy-plane. Thus distances will be preserved.

Exercises 12-1

1. Draw a pair of perpendicular lines. Call the intersection the point O and lay off common scales on the two axes. Plot the points whose coordinates are $(-2,3)$, $(4,1)$, $(\frac{1}{2},0)$, $(0,-3)$ and $(-\pi,-\pi)$. It should be clear from the plotting process that a unique point is determined by any particular ordered pair of numbers.

2. In the figure to the right what are the coordinates of P, Q, R, S, and T? What are the coordinates of O? (We will have to estimate coordinates that are not clearly whole numbers).



3. (a) The IVth quadrant is the set of all points (a,b) for which a $\geq$ 0 and b ? 0.

   (b) Make similar statements about the I, II, and III quadrants.

4. (a) What is the set of all points with x-coordinate negative?

   (b) What is the set of all points with y-coordinate greater than or equal to zero?

5. (a) What is the set of all points with x-coordinate equal to 0?

   (b) What is the set of all points with x and y-coordinates both zero?

   (c) What is the set of all points with at least one coordinate zero?

---

2. Graphs of Algebraic Statements or Sentences.

Consider any statement about a number x and a number y. Examples of such statements are $x + y = 10$, $x > y$, $x = 2$ (this qualifies as such a statement because it says that x is 2 and specifically does not restrict y), $y > -1$, $y = 3 + 2x$ and $y = x^2$. Frequently, but not always, the statement is an equation or an inequality. We call such a statement an algebraic statement about x and y.

Definition: The graph of an algebraic statement about x and y is the set of all points (in the plane) whose coordinates make the statement true (satisfy the statement).

This is a very important definition. It is the key relationship between algebra and geometry (between algebraic statements or sentences and sets of points). The formulation and cultivation of the point of view leading to this relationship between algebra and geometry is credited to the French philosopher and mathematician, Rene Descartes. It is probably one of the most significant scientific contributions ever made. Today we still speak of rectangular coordinates (as in Section 1) as Cartesian coordinates.

There are three main types of problems about graphs.

(1) Given an algebraic statement what can be said about its graph?

(2) Given an algebraic statement, draw its graph.

(3) Given a set of points, what is an algebraic statement of which it is the graph?

We can give answers to these questions in many simple cases. In answering the question as to what can be said about the graph of an algebraic statement we desire an answer in set language; i.e., a description of a set of points. In (2) we desire an actual picture or drawing of the graph where possible.

Note that there are two considerations in deciding whether a particular set M of points is the graph of an algebraic statement.

(a) Do the coordinates of every point in the set M make the algebraic statement true?

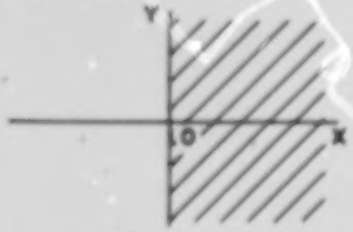
(b) Is every point whose coordinates make the algebraic statement true in the particular set M of points?

Let us consider a few elementary examples.

(1) $x > 0$. The graph is, by definition, the set of all points for which x is positive. This will be the set of all points to the right of the y-axis.

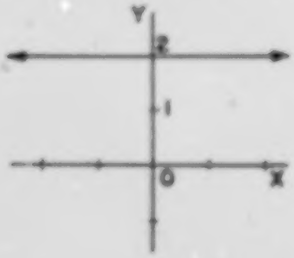
(a) Any point to the right of the y-axis has the property that its x-coordinate is positive.

(b) Any point whose x-coordinate is positive must be to the right of the y-axis.



(2) $y = 2$. The graph is, by definition, the set of all points for which $y = 2$.

The graph is the line two units above the x-axis. Let us see why.



(a) Every point of that line has the property that $y = 2$.

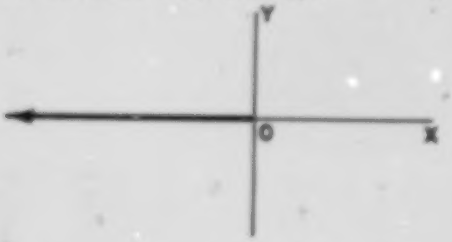
(b) Every point whose y-coordinate is 2 (which makes $y = 2$ a true statement) is on that line.

(3) $x < 0$ and $y = 0$. The graph of this statement is, by definition, the set of all points for which x is negative and y is zero. Let us see what the graph must be.

The set of points for which x is less than zero is the set of all points to the left of the y-axis. Call this set H. The set of points for which $y = 0$ is the x-axis. Call this set K. The graph we seek is the set of all points which are in H and are also in K; i.e., the set of points of $H \cap K$. This set is clearly the set of points of the x-axis which are to the left of the y-axis. Thus we have described the graph for

(a) Every point in this set $(H \cap K)$ has coordinates satisfying the algebraic statement and

(b) Every point whose coordinates satisfy the statement is in this set $(H \cap K)$.

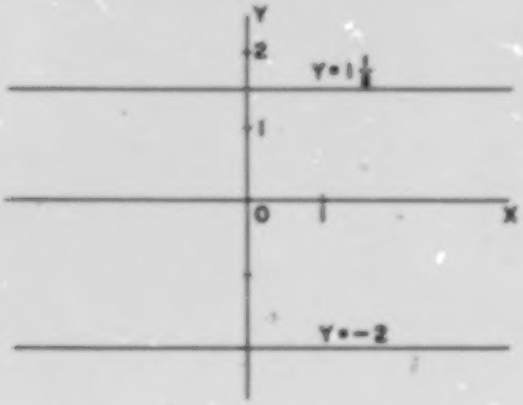


(4) $x = a$, for a any particular real number. Examples are $x = 1$, $x = -\pi$, $x = 6 - \sqrt{2}$, etc.

Any point P whose coordinates make the statement $x = a$ true is a point whose coordinates are of the form (a,y). Graphically, it is a point whose projection on the x-axis is the point of the axis whose x-coordinate is a. Therefore the graph we seek is the set of all points on the line perpendicular to the x-axis and "a" units away from the y-axis. If $a > 0$, the line is to the right of the y-axis. If $a < 0$, it is to the left of the y-axis. If $a = 0$, it is the y-axis.

(5) $y = b$ for b any particular real number.

From reasoning like that above, the graph must be a horizontal line, b units from the x-axis, above, on, or below as b is positive, zero, or negative respectively.



Exercises 12-2

Graph the following algebraic statements:

1. $x > 1$
2. $y = x$
3. $x = -1$ and $y = 2$
4. $x = -1$ or $y = 2$
5. $x \cdot y > 0$
6. $y < 2$ and $x > 0$
7. $x \cdot y = 0$
8. $x \cdot y \neq 0$

Give algebraic statements of which the following are descriptions of their graphs.

9. The set of points to the left of the y-axis.

10. The set of points not in the union of the II, III, and IVth quadrants.

11. The origin.

---

3. Graphing Techniques.

The traditional elementary way to graph an algebraic statement which is an equation has been to "plot points". Consider the equation $y = 1 + x^2$, for instance. We would compile a table as follows:

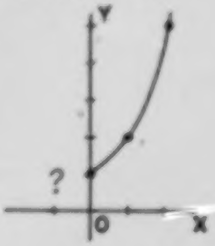
x	y
0	1
1	2
2	5

When $x = 0$ then $y = 1 + 0 = 1$

when $x = 1$ then $y = 1 + 1^2 = 2$

when $x = 2$ then $y = 1 + 2^2 = 5$

etc.



Then we would graph the points (0,1),(1,2) and (2,5) and possibly some others and "guess" at what other points might be on the graph. In easy examples (like the above) we were usually right. But certainly the "point plotting" method leaves much to be desired. It does not answer our fundamental questions (a) and (b) of the preceding section about the graph.

Consider the equation $xy = 6$.

Let us plot points. For $x = 0$, it doesn't work

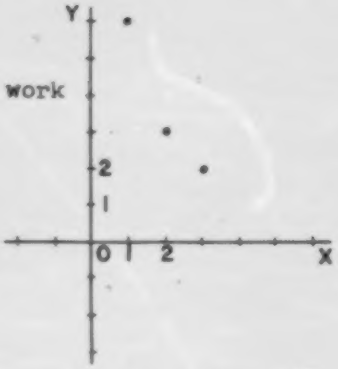
x	y
1	6
2	3
3	2
6	1
-2	-3

for $x = 1$, $y = 6$

for $x = 2$, $2y = 6$

$y = 3$

etc.



We plot the five points whose coordinates are given above. Now how do we draw the graph? It is not easy or obvious simply from these considerations.

So let us start over again and try to collect information which will let us be reasonably sure that what we will draw will look like the graph ought to look. We seek answers to some or all of the following questions. The answers themselves are not important. It is the use to which we put the answers that is important. In a given problem, we answer the "easy" questions first and see if we then have enough information to help us graph the equation.

(1) Is the equation (or statement) of a type for which we already know what the graph must be? If so, graph it and use the other questions only as a check. For instance, if the equation is $x = 3$ we know what the graph must be. It is the vertical line 3 units to the right of the y-axis.

(2) For what values of x is there a corresponding value of y? (What is the set of all numbers "a" for which the graph contains a point with first coordinate "a"?)

Consider $y = 3 + 2x$	Consider $xy = 6$
In this equation, it is clear that for any value of x there will be a corresponding value of y. We can see this by just looking at the equation. Think of substituting a number for x; then y is 3 plus twice that number.	In this equation, it is clear that if $x = 0$ then there is no corresponding value of y. If $x \neq 0$ then there is a corresponding value of y (for we can then solve for y).

What do these observations mean graphically? They mean that for any value of x for which there is at least one corresponding value of y, there will be at least one point of the graph on the vertical line determined by that value of x. By the same token, if there is no corresponding value of y for a particular value of x, then the graph can not contain any point on such vertical line.

Consider $y = 3 + 2x$.	Consider $xy = 6$
The graph contains at least one point on each vertical line.	The graph contains no point on the y-axis (the line $x = 0$). The graph contains at least one point on each other vertical line.

(2') The same as (2), but with the roles of x and y reversed.

(3) For a given value of x, how many corresponding values of y are there?

In both of our examples, there was never more than one corresponding value of y for any value of x. Graphically, this means, for our examples, that *neither graph contains two points on any vertical line*. (An equation like $y^2 = x^2 + 1$ would have two points on each vertical line. For $x = 0$, for instance, y could be either +1 or -1.)

(3') The same as (3) but with the roles of x and y reversed.

(4) For what values of x is $y > 0$? is $y < 0$? For what values of y is $x > 0$? is $x < 0$?

Consider $y = 3 + 2x$.

$y > 0$ whenever $3 + 2x > 0$

or $2x > -3$

or $x > -\frac{3}{2}$.

$y < 0$ whenever $x < -\frac{3}{2}$.

This means that the graph is above the x-axis for $x > -\frac{3}{2}$ and is below the x-axis for $x < -\frac{3}{2}$.

Consider $xy = 6$

$y > 0$ whenever $x > 0$

$y < 0$ whenever $x < 0$.

This means that the graph is contained in quadrants I and III.

(5) If relevant, how large is y if x is a large number? How large is x if y is a large number?

Consider $y = 3 + 2x$.

If x is large, y is large, it is 3 plus twice x.

If y is large, x must also be large (about half as large as y).

Consider $xy = 6$.

If x is large, y must be small, in fact, close to zero.

If y is large, x must be close to zero. For instance, if $x = 100$, $y = \frac{6}{100}$.)

Having collected information in answering some or all of these questions, we then have the problem of actually graphing the equation consistent with what we have learned.

Finally in actually doing the graphing, we usually do plot some points. Then we draw the graph through these points on the basis of the other information we have gathered. Having drawn the graph, we should then check to see that it conforms to our information.

Exercises 12-3

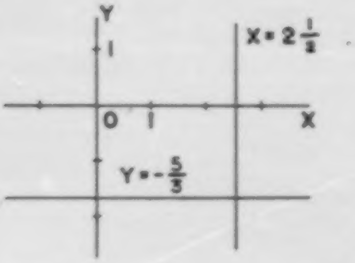
1. Using the discussion in the text, graph $y = 3 + 2x$.
2. Using the discussion in the text, graph $xy = 6$.

Discuss (with respect to our 5 questions) and graph,

3. $y = 1 + x^2$
4. $y = 2 - x$
5. $y = x^3$
6. $xy = -12$

4. Linear Equations.

We have already noted in Section 2 that an equation like $x = a$ (or $y = b$) has a graph which is a (straight) line.

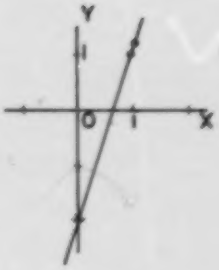


In the figure to the left $a = 2\frac{1}{2}$ and the graph of $x = 2\frac{1}{2}$ is the vertical line indicated. Similarly $b = -\frac{5}{3}$ and the graph of $y = -\frac{5}{3}$ is the horizontal line indicated.

There are other equations which have graphs which are (straight) lines. In fact any equation of the form $y = mx + b$ has a graph which is a straight line. An example is $y = 5x - 12$. From the considerations of Section 3, even without knowing that the graph is a straight line, we can immediately conclude that the graph must cross each vertical line exactly once and if $m \neq 0$ it must also cross each horizontal line exactly once.

Clearly the graph of the equation $y = mx + b$ passes through the point $(0,b)$ for $b = m \cdot 0 + b$ and thus $(0,b)$ satisfies the equation. Also the graph has slope m; i.e., if you increase x by k units you increase y by $m \cdot k$ units. We explain this idea by an example.

Let us consider $y = 3x - 2$. The point $(0, -2)$ is on the graph. If x is increased from 0 to 1, y is increased by 3. If x is increased from 1 to 1.1 then y is increased by $3 \cdot \frac{1}{10}$ or .3.



A proof that the graph of $y = mx + b$ is actually a straight line depends on equality of ratios of corresponding sides of similar triangles. We do not give the details here.

Using the information above, we can prove that any line must have an equation of the form $y = mx + b$ or $x = a$. If the line is vertical an equation of the line is of the form $x = a$. If the line is not vertical then the line must intersect the y-axis at a point whose y-coordinate we will call b. The line must intersect the line $x = 1$ at a point whose y-coordinate we call d. Now $d - b$ is the increase in y when x is increased from 0 to 1. The line whose equation is $y = (d - b) \cdot x + b$ does pass through two points on our given line, namely $(0,b)$ and $(1,d)$. Therefore our given line and the line whose equation is $y = (d - b) \cdot x + b$ must be identical. Hence $y = (d - b) \cdot x + b$ is an equation of our line.

Thus we have shown that every line has an equation of the form $y = mx + b$ or of the form $x = a$.

Linear Equations. An equation in x and y is said to be of the first degree in x and y if it can be put in the form $Ax + By + C = 0$ where at least one of the numbers A and B is not zero. We note that $x = a$ is of this form for $1 \cdot x + 0 \cdot y + (-a) = 0$ is equivalent to $x - a = 0$ and hence to $x = a$. Note that $A = 1$ and hence $A \neq 0$.

We also note that $y = mx + b$ is of this form for $(-m)x + 1 \cdot y + (-b) = 0$ is equivalent to $-mx + y - b = 0$ and hence to $y = mx + b$. Note that $B = 1$ and hence $B \neq 0$. Thus we have shown that every line is the graph of an equation of the first degree in x and y (for every line is a graph of an equation of the form $y = mx + b$ or $x = a$).

Let us look at the other side of the coin. Is it true that every equation of the first degree in x and y has a graph which is a (straight) line? The answer is "yes" and we proceed to prove the assertion based on our earlier observations.

Consider $Ax + By + C = 0$ with at least one of A and B not zero. Suppose $B \neq 0$. Then it follows from elementary properties of numbers that the equations $Ax + By + C = 0$

$$By = -Ax - C$$

and

$$y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)$$

are equivalent. (We say that such equations are equivalent if they have the same solutions; i.e., provided that if any ordered pair of numbers (x,y) satisfies one equation it also must satisfy the other(s).) If the equations are equivalent they must have the same graph. Thus if $B \neq 0$ the graph of $Ax + By + C = 0$ is the graph of $y = (\frac{-A}{B})x + (\frac{-C}{B})$ and we have already agreed that the latter graph is a (straight) line. (We consider $(\frac{-A}{B})$ to be m and $(\frac{-C}{B})$ to be b).

Finally we ask what the situation is if $B = 0$. Then $A \neq 0$ (for at least one of A and B is not zero) and the equations

$$Ax + C = 0,$$

$$Ax = -C,$$

and

$$x = (\frac{-C}{A})$$

are equivalent. But the graph of $x = (\frac{-C}{A})$ is known to be a vertical line. Thus the graph of $Ax + By + C = 0$ is a line provided at least one of A and B is not zero.

We call an equation of the first degree in x and y a *linear equation* because its graph is a line.

Whenever an equation is given which is equivalent to an equation of the form $Ax + By + C = 0$ (A or B not zero) we know the graph must be a (straight) line. We can graph the equation by finding two points on the line and using a straight edge or ruler to draw the line. (We frequently find a third point just to check our arithmetic.)

Important Conclusion.

Finally we can well ask what the significance of this point of view is. It is monumental.

Much of algebra is a study of linear equations in x and y. Much of geometry is a study of (straight) lines.

When we study either (a) properties of lines or sets of lines in geometry or (b) properties of linear equations or sets of linear equations in algebra we are really studying both. We can learn about linear equations by thinking about lines. We can learn about lines by thinking about linear equations.

Exercises 12-4

1. What is the graph of $Ax + By + C = 0$ if A and B are both zero and $C \neq 0$?
2. What is the graph of $Ax + By + C = 0$ if A, B, and C are each zero?
3. Graph $y = 2x - 1$
4. Graph $y = (-1)x + 3$
5. Graph $3x - 2y = 6$
6. Graph $2x + 4y = 1$

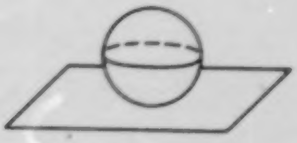
Chapter 13

The Sphere

1. Properties.

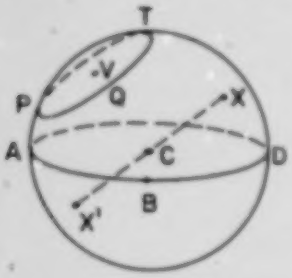
The ordinary mathematical abstraction of the surface of a round ball is called a sphere (or a "2-dimensional sphere" in some contexts). The sphere is also used as a mathematical abstraction of the surface of the earth. The fact that the surface of the earth is somewhat uneven and is thought to be a bit flattened at the poles is, from many points of view, not important. It is still useful to study the sphere and to regard it as an abstraction of the surface of our earth. A sphere like a circle has a center. In fact, given a positive number R and a point C, the set of all points of space at distance R from C is called the sphere of radius R and center C.

Consider the intersection of a plane and a sphere. If the intersection is not empty then it might be just one point. In such case the sphere would be tangent to the plane. This situation would be represented by a hard ball resting on a table. The surface of the ball seems to have just one point in common with the table top.



If the intersection of a plane and a sphere is not empty and contains more than one point, then it is a circle. One sees an illustration of this by a slicing of an orange.

There is a distinction made as to whether the plane which intersects the sphere contains the center of the sphere. If it does, we call the intersection a great circle of the sphere. If the plane does not contain the center then we call the intersection a small circle. Note that the center of the sphere is also the center of each of the great circles of the sphere but it is not the center of any of the small circles of the sphere.



In the figure, PQT represents a small circle with center at V. ABD represents a great circle with center at C, the center of the sphere.

Given any point X on the sphere, there is exactly one line in space containing X and the center C. This line must also intersect the sphere at exactly one other point. Call it X'. (We read it "X-prime".) Then X and X' are the endpoints of a diameter of the sphere and are called diametrically opposite points.

The north and south poles represent diametrically opposite points on the surface of the earth. The equator represents a great circle. Let us note two fundamental properties of a great circle.

Property I. *Every two distinct great circles on a sphere have a non-empty intersection and the intersection is a set of two points which are diametrically opposite.*

Proof: Each great circle is the intersection of the sphere and a plane which contains the center of the sphere. The two distinct planes which contain the great circles have the center of the sphere in common. Therefore, their intersection is a line which contains the center of the sphere. But this line which contains the center of the sphere must intersect the sphere in exactly two points which are diametrically opposite. The intersection of the two great circles is precisely the intersection of the sphere and the set which is the intersection of the two planes. Hence, the intersection of two distinct great circles is a set of two points which are diametrically opposite.

Property II. *If A and B are any two distinct points of a sphere and A and B are not diametrically opposite, then there is exactly one great circle of the sphere containing A and B.*

Proof: A, B, and the center C of the sphere are not on the same straight line (because A and B are not diametrically opposite). Therefore, from Property III of Chapter 5 there is a unique plane containing A, B, and C. But because this plane contains C, it must intersect the sphere in a great circle and such great circle must contain A and B.

If any other great circle contained A and B, the plane containing that great circle would also contain C (of course) and we would have two distinct planes containing A, B, and C. This is impossible and thus Property II is proved.

One of the interesting and important facts about spheres is that if A and B are two points of a sphere then the shortest path on the sphere between A and B is the great circle path from A to B. This fact is of great significance in navigation, both in ship sailing routes and in airline routes.

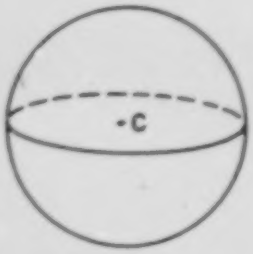
We may experimentally anticipate this result by taking a globe and stretching a string between two points on it.

Exercises 13-1

1. (a) Make a drawing of a sphere like that on the right.

   (b) Label 4 points of the equator in diametrically opposite pairs.

   (c) Dot in the segments joining the diametrically opposite pairs in (b).



   (d) Draw two small circles, one of which intersects the equator and one of which doesn't. Label their centers.

2. Draw a sphere and two great circles on the sphere showing their points of intersection to be diametrically opposite.

3. Draw a sphere with its equator. Draw four small circles of the sphere each in a plane parallel to the plane of the equator.
4. Take a round ball or globe and stretch a string between two points on it to check the "shortest distance" fact about spheres. Try this several times to help your intuition.
5. Take an orange or an apple and slice it to show great circles and small circles.
6. Explain why going due north would be the most efficient way of getting to a point due north of your starting point.
7. (a) Explain why going due east is usually not the most efficient way of getting to a point which is due east of your starting point.
   (b) Describe special circumstances when it would be the most efficient.

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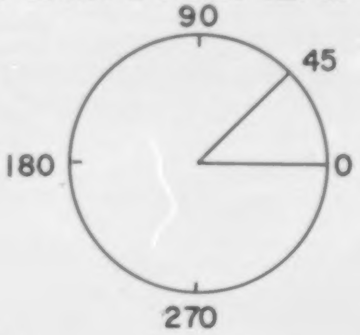
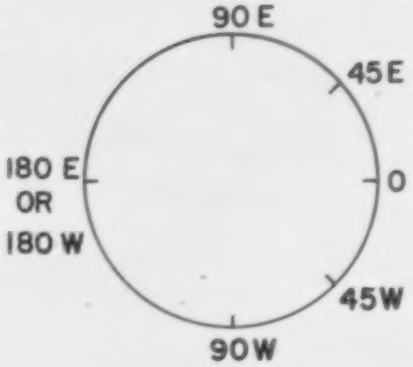
2. Coordinatization of the Sphere.

We have seen in Chapter 12 how we could coordinatize the plane. Given two perpendicular reference lines as the axes, we could locate any point by knowing the x- and y-coordinates of the point.

How do we coordinatize the surface of the earth--a sphere? Our ancestors set up a coordinate system. They were aided by knowledge of the earth's rotation. The earth, of course, is considered to rotate on an axis--the line containing the north and south poles and the center of the earth. The set of points halfway between the north and south poles and on the surface is called the equator. It turns out that this set is a great circle. It is reasonable to use the poles and the equator as reference sets in our coordinate system. We call the great semi-circles which have the north and south poles as endpoints the *meridians*. As each great circle containing the poles intersects the equator in two diametrically opposite points each meridian intersects the equator in a unique point. There is a one-to-one correspondence between the set of meridians and the set of points of the equator. Each point of the equator corresponds to the meridian which contains it. Furthermore, except for the two poles, each point of the sphere is on exactly one meridian. Thus if we coordinatize the set of meridians we can use this coordinate to help locate the point. Note, too, that if we coordinatize the equator we can consider the set of meridians to be coordinatized by use of the one-to-one correspondence of the set of points of the equator with the set of meridians.

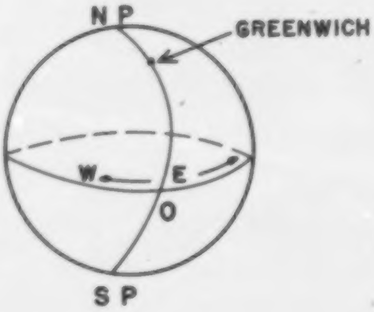
We have already seen in Chapter 11 how a circle can be coordinatized in units of degree measure. There are several options in some details of how we choose to do such. We can use numbers from 0 to 360 using a counter-clockwise system.



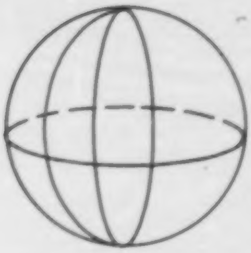


Or we can choose to measure from our 0 point both ways to 180, one direction being positive and the other negative (or what is more convenient for the equator on the earth, one east and one west). We call the coordinate of the meridian on which a point lies the longitude of the point.

Many years ago it was decided to call the Greenwich meridian the zero (or prime) meridian. The Greenwich meridian is that one which passes through a particular point of the town of Greenwich, England. The rest of the meridians are numbered east or west of the Greenwich meridian. If we think of looking down at the equator from the north pole then we would label points of the equator as in the figure above. The 180th meridian runs north and south through the Pacific Ocean and the eastern tip of Siberia. It is used for much of its extent as the so-called International Date Line.



Now to locate a point on a sphere if we have poles and meridians selected we need to know both what meridian the point is



on and how far above or below the equator the point is. The "natural" way to measure "distance" above or below the equator is in terms of arc length on the meridian. And this is what is customarily done.
The portion of a meridian from the equator to either pole is a quarter of a circle. If the point on the equator is identified as the zero point on this quarter circle then the pole would be a 90 (degree) point and each other point would have a coordinate (called its latitude) between 0 and 90 and north or south as the pole is north or south. The set of all points with latitude equal to say 45 north is a small circle on the sphere. The plane containing this circle is parallel to the plane of the equator --hence the expression "parallels of latitude".

The north and south poles, equator, longitude and latitude coordinatization of the sphere is used by mathematicians in many contexts quite apart from those related to the surface of the earth. It just happens to be the case that this system is about as simple, convenient, and useful as any that can be set up.

One of the interesting aspects of our coordinatization of the sphere is that, except at the poles, "locally" it is similar to the coordinatization of a plane. What we mean by "locally" is that one can choose to think of only a small portion of the sphere. Then the meridians are like vertical lines and the parallels of latitude are like horizontal lines.

Exercises 13-2

1. Describe the set of points of the sphere which have exactly two different longitudes (as we have described it).
2. What is the set of points of the sphere each of which has more than two longitudes?
3. What is the set of points of the sphere which have more than one latitude?
4. Draw a sphere with an equator and with a meridian to represent the Greenwich meridian. On your drawing label the following points:

   P: (0 E, 85 N)
   Q: (45 W, 10 S)
   R: (90 W, 90 S)
   S: (180 E, 0 N)
   T: (25 E, 25 N)

5. Consider a different coordinatization of the sphere as follows:

   The set of meridians is to be coordinatized as before.

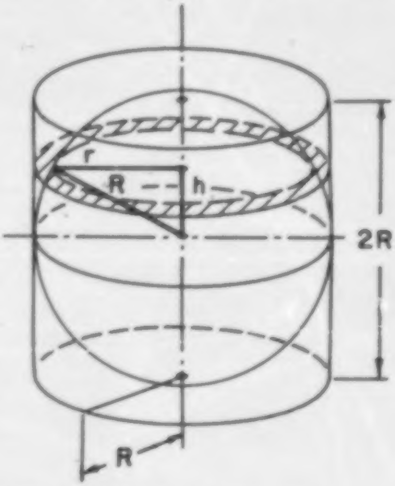
   The parallels of latitude are to be numbered starting from the south pole as zero, with the north pole as 180 and using arc length along meridians from the south pole.

   Every point of the equator would have "latitude" ______. Every point in the northern hemisphere would have "latitude" greater than ______ and less than ______.

3. The Volume of a Spherical Ball and the Area of a Sphere.

In this section we try to give some understanding of the formulas for volume and surface area of a sphere. As in other contexts the volume of a sphere refers to the volume of the portion (region) of space bounded by the sphere. The surface area of the sphere is the area of the sphere itself. In terms of practical problems, the volume can be regarded as the amount of sand it would take to fill up a spherical ball whereas the area can be regarded as the amount of surface to be covered in painting the sphere.

We develop the volume formula first. From it we shall get the surface area formula. Let us think of a sphere contained in the interior of a cylinder which just fits around it. Let R be the radius of the sphere. Then the height of the cylinder is 2R and the radius of the base is R. Let V_S be the volume of the sphere and V_C the volume of the cylinder. Thus $V_S < V_C$ and we expect V_S to be considerably less than V_C. In Chapter 11, we have developed the formula

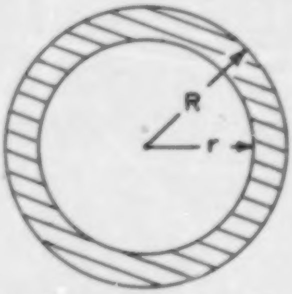


$$V_C = B \cdot H = (\pi R^2)(2R) = 2\pi R^3.$$

Therefore the volume of the top half of the cylinder is πR^3. We seek the volume of the top half of the sphere--i.e., of the northern hemisphere. The volume of the sphere is twice that of the top half.

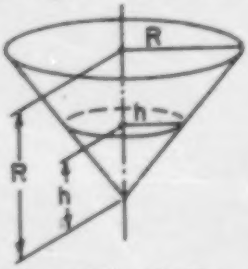
Think of a plane parallel to the base of the cylinder which cuts through the cylinder and the sphere at a distance of h units above the equator. Then the area of the circular region cut out by the cylinder is πR^2. The area of the circular region cut out by the sphere is πr^2 (if r is the radius of the small circle on the sphere). (See the triangle in the figure on the preceding page.) But $r^2 + h^2 = R^2$ by the Pythagorean Theorem. Hence $r^2 = R^2 - h^2$. Therefore the area of the larger circular region minus the area of the smaller is

$$\begin{aligned}\pi R^2 - \pi r^2 &= \pi R^2 - \pi(R^2 - h^2)\\ &= \pi R^2 - \pi R^2 + \pi h^2\\ &= \pi h^2\end{aligned}$$



The shaded region represents the region of the plane inside the cylinder and outside the sphere.

Now consider a cone (upside down) whose base is a circular region of radius R and whose height is R. The area of the plane section of this cone h units above the vertex is πh^2 since the radius of the circular section at that level is h.



This means that the cross section area of the part of the cylinder not in the sphere is exactly the cross section area of a cone as described above. Therefore it is reasonable to believe that the volume of the top half of the cylinder minus the volume of the top half of the sphere is exactly the volume of the inverted cone (for the horizontal plane sections have the correct areas). Note that while we do not actually have a cone in the figure with the sphere inside the cylinder, we have an object (odd-shaped) whose volume is the same as the volume of the cone we have considered.

The volume of a cone is $\frac{1}{3}$ the area of the base times the height. Hence the volume of the cone (V_{cone}) is $\frac{1}{3}(\pi R^2)\cdot R$. $V = \frac{1}{3}\pi R^3$. Also $V_S = V_C - 2V_{cone}$ (for we have two cones to be considered, one for the top half and one for the bottom half of the cylinder). Hence $V_S = 2\pi R^3 - 2 \cdot \frac{1}{3}\pi R^3 = \frac{4}{3}\pi R^3$, which is the usual formula.

This is a valid formula for the volume of a sphere of radius R. Now we are in a position to justify the formula for the surface area of a sphere. Suppose we wish to find the volume of rubber in a rubber ball which is hollow inside and which has only a thin rubber coating. The volume of the spherical shell is the volume of the outside sphere minus the volume of the inside sphere (the volume of the inside sphere is the volume of the void in the middle). Let r_2 be the radius of the outside sphere and r_1 be the radius of the inside. Let $V_{S.S.}$ be the volume of the spherical shell. Then

$$\begin{aligned} V_{S.S.} &= \frac{4}{3}\pi r_2^{\,3} - \frac{4}{3}\pi r_1^{\,3} \\ &= \frac{4}{3}\pi\,(r_2^{\,3} - r_1^{\,3}) \\ &= \frac{4}{3}\pi\,(r_2 - r_1)(r_2^{\,2} + r_1 r_2 + r_1^{\,2}). \end{aligned}$$

This last formula follows because

$$(r_2 - r_1)(r_2^2 + r_1r_2 + r_1^2) = r_2^3 - r_1^3$$

as may be seen by multiplying the two factors on the left together. But $(r_2 - r_1)$ is simply the thickness of the shell, i.e., the thickness of the rubber coating. It is the outside radius minus the inside radius. If r_2 is close to r_1 (i.e., if we have a thin shell) then the volume of the spherical shell would seem to be almost the surface area A_S of the outside sphere times the thickness of the shell.

Therefore we now have

$$V_{S.S.} = \frac{4}{3}\pi\,(r_2 - r_1)(r_2^2 + r_1r_2 + r_2^2)$$

and

$V_{S.S.} \approx A_S \cdot (r_2 - r_1)$ where $\approx$ means "is approximately equal to".

Therefore $A_S \approx \frac{4}{3}\pi\,(r_2^2 + r_1r_2 + r_2^2)$ provided r_1 and r_2 are close together. But if r_1 and r_2 are close together, then r_2^2 and r_1^2 are close together and r_2^2 $(= r_2 \cdot r_2)$ and r_1r_2 are close together.

Thus $$r_2^2 + r_1r_2 + r_1^2 \approx 3r_2^2.$$

Hence $$A_S \approx \frac{4}{3}\pi\,(3r_2^2)$$
$$\approx 4\pi r_2^2.$$

But r_2 is the radius of the sphere, hence $A_S \approx 4\pi r^2$ and the approximation can be made as close as we want. Thus it turns out that $A_S = 4\pi r^2$. This is the usual formula for the surface area of a sphere.

Exercises 13-3

1. Find the volume of a spherical grapefruit whose "distance around the middle" is 18".
2. Find the amount of paint needed to paint the outside of a spherical tank 20' in diameter if one gallon of paint will cover 400 square feet.
3. Find the volume of rubber needed to make 1000 hollow rubber balls of outside diameter 3" if the thickness of the rubber in each ball is to be .1".
4. Three tennis balls just fit in a cylindrical can designed to hold them, one above the other. Find the volume of the air space left in a can full of three balls if the radius of a ball is about 1.3".

Chapter 14

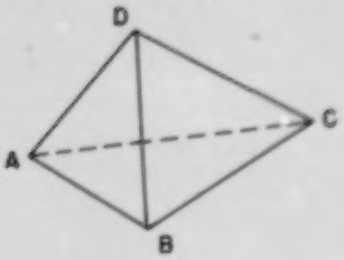
Non-Metric Polyhedrons

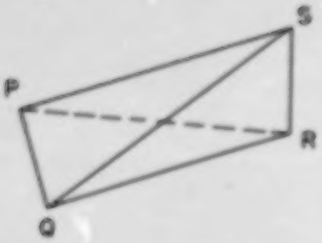
The material of this chapter will be new to almost all people who are studying it in this text. Most of it has been tried (in much its present form) in several eighth grade classes with rather surprising success. There are a number of reasons for including it in the eighth grade curriculum. Among these are:

1. It helps develop spatial intuition and understanding.
2. It emphasizes in another context the role of mathematics in reducing things to their simplest elements.
3. It affords other ways of looking at objects in the world about us and raises fundamental questions about these.
4. It illustrates types of mathematical (geometric) reasoning and approaches to problems.
5. It gives an interesting insight into the meaning of dimension.

1. Tetrahedrons and Simplexes.

A geometric figure of a certain type is called a tetrahedron. A tetrahedron has four vertices which are points in space. The drawings below represent tetrahedrons. (Another form of the word "tetrahedrons" is "tetrahedra").



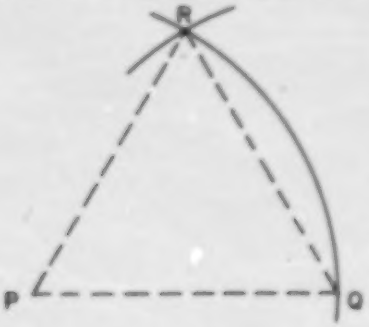


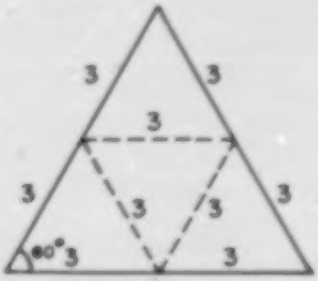
The points A, B, C, and D are the vertices of the tetrahedron on the left. The points P, Q, R, and S are the vertices of the one on the right. The four vertices of a tetrahedron are not in the same plane. The word "tetrahedron" refers either to the surface of the figure or to the "solid" figure; i.e., the figure including the interior in space. From some points of view, the distinction is not important. Later we shall use the term "solid tetrahedron" when we mean the surface together with the interior. We can name a tetrahedron by naming its vertices. We shall normally put parentheses around the letters like (ABCD) or (PQRS). Later we shall use this notation to mean "solid tetrahedron".

The segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$, $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ are called the edges of the tetrahedron (ABCD). We sometimes will use the notation (AB) or (BA) to mean the edge $\overline{AB}$. What are the edges of the tetrahedron (PQRS)?

Any three vertices of a tetrahedron are the vertices of a triangle and lie in a plane. A triangle has an interior in the plane in which its vertices lie (and in which it lies). Let us use (ABC) to mean the triangle ABC together with its interior. In other words, (ABC) is the union of △ABC and its interior. The sets (ABC), (ABD), (ACD), and (BCD) are called the faces of the tetrahedron (ABCD). What are the faces of the tetrahedron (PQRS)?

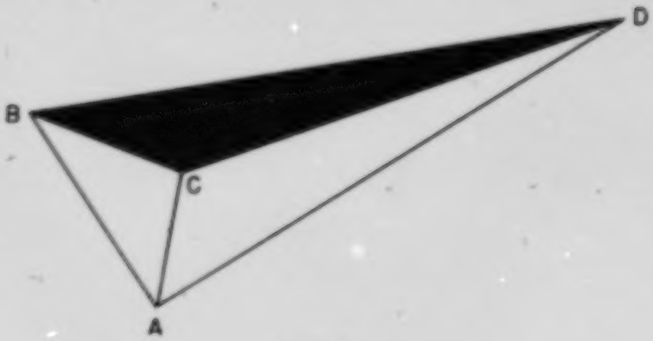
You will be asked to make some models of tetrahedrons in the exercises. In teaching material like this to junior high school students, the models are likely to be considerably important. Prior awareness of and facility with models should increase teaching effectiveness as well as improve basic understandings. The easiest type of tetrahedron of which to make a model is the so-called regular tetrahedron. Its edges are all the same length. (We introduce length or measurement here only for convenience in making some uniform models. This chapter deals fundamentally with non-metric or "no-measurement" geometry.) On a piece of cardboard or stiff paper construct an equilateral triangle of side 6". (You can do this with a ruler and compass or with a rulerand protractor.)





Now mark the three points that are halfway between the various pairs of vertices. Cut out the large triangular region. Carefully make three folds or creases along the segments joining the "half-way" points. You may use a ruler or other straightedge to help you make these folds. Your original triangular region now looks like four smaller triangular regions. Bring the original three vertices together above the center of the middle triangle. Fasten the loose edges together with tape or paper and paste. You now have a model of a regular tetrahedron.

How do we make a model of a tetrahedron which is not a regular one? Cut any triangular region out of cardboard or heavy paper. Use this as the base of your model. Label its vertices A, B, and C. Cut out another triangle with one of its edges the same length as $\overline{AB}$. Now, with tape, fasten these two triangles together along edges of equal length. Use edge (AB) for this, for instance. Two of the vertices of the second triangle are now considered labeled A and B. Label the other vertex of the second triangle D. Cut out a third triangular region with one edge the length of $\overline{AD}$ and another the length of $\overline{AC}$. Do not make the angle between these edges too large or too small. Now, with tape, fasten these edges of the third triangle to $\overline{AD}$ and $\overline{AC}$ so that the three triangles fit together in space. The model you have constructed so far will look something like a conical drinking cup if you hold the vertex A at the bottom. Finally cut out a triangular region which will just fit the top, fasten it to the top and you will have your tetrahedron.

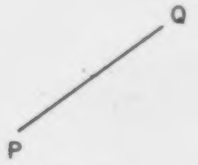


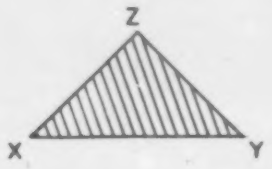
Exercises 14-1a

1. Make a cardboard or heavy paper model of a regular tetrahedron. Make your model so that its edges are each 3" long.
2. Make a model of a tetrahedron which is not regular.
3. In making the third face of a non-regular tetrahedron, what difficulties would we encounter if we made the angle DAC too large or too small?

Simplexes. A single point is probably the simplest object or set of points you can think of. A set consisting of two points is probably the next most simple set of points in space. But any two different points in space are on exactly one line and are the endpoints of exactly one segment (which is a subset of the line). Thus, the set of two points determines two other simple sets in space: a line and a segment. A segment has length but does not have area. We speak of a segment or a line as being one-dimensional. Either could be considered as the simplest one-dimensional object in space. In this chapter we want to think about the segment, not the line.

A set consisting of three points is the next most simple set of points in space. What do three points in space determine? If the three points are all on the same line, then we get just a part of a line. We are not much better off than we were with just two points. Let us agree, therefore, that our three points are not to be on the same line. Thus there is exactly one plane containing the three points and there is exactly one triangle with the three points as vertices. There is also exactly one triangular region which together with the triangle which bounds it, has the three points as vertices. This mathematical object, the triangle, together with its interior, is what we want to think about. It has area and it is two-dimensional. It can be considered as the simplest two-dimensional object in space.





It seems rather clear that the next most simple set of points in space would be a set of four points. If the four points were all in one plane then the figure determined by the four points would apparently also be in one plane. We want to require that the four points are not all in any one plane. This requirement also guarantees us that no three can be on a line. (If any three were on a line then there would be a plane containing that line and the fourth point and the four points would be in the same plane.) We have four points in space, then, not all in the same plane. Clearly, this suggests a tetrahedron. The four points in space are the vertices of exactly one (solid) tetrahedron. A solid tetrahedron has volume and it is three-dimensional. It can be considered as the simplest three-dimensional object in space.

Here we have four objects each of which may be thought of as the simplest of its kind. There are remarkable similarities among these objects. They all ought to have names that sound alike and remind us of their basic properties. We call each of these a simplex. We tell them apart by labeling each with its natural dimension. Thus a set consisting of a single point is called a 0-simplex. A segment is called a 1-simplex. A triangle together with its interior is called a 2-simplex. A solid tetrahedron is called a 3 simplex.

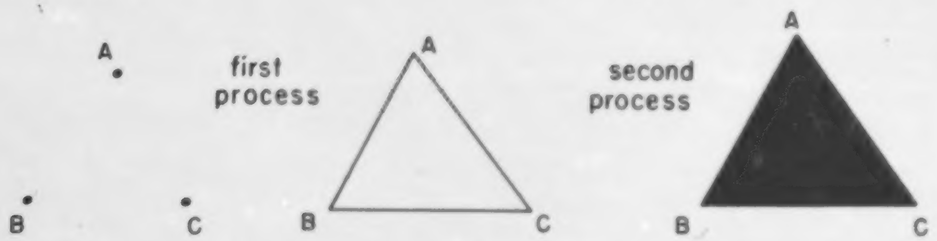
Let us make up a table to help us keep these ideas in order.

A set consisting of:	determines:	which is called a:
one point	one point (itself)	0-simplex
two points	a segment	1-simplex
three points not all on any one line	a triangle together with its interior	2-simplex
four points not all on any one plane	a solid tetrahedron (which includes its interior)	3-simplex

There is another way to think about the dimension of these sets. In this we think of the notion of betweenness, of a point being between two other points.

Let us start with two points. Consider these two points and all points between them. We now have a segment. Now take the segment *together with* all points which are between *any* two points of the segment. We still have the same segment. No new points were obtained by "taking points between" again. The process of "taking points between" needed to be used just once. We get a one-dimensional set, a 1-simplex.

Next consider three points not all on the same line. Then let us apply our process. We take these points together with all points which are between any two of them. At this stage we have a triangle but not its interior. We apply the process again. We take the set we already have (the triangle) *together with* all points which are between *any* two points of this set. We get the union of the triangle and its interior. If we apply the process again we don't get anything new. We need use the process just twice. We get a two-dimensional set, a 2-simplex.



Next let us consider four points not all on the same plane. We apply the process of "taking points between" and get the union of the edges of a tetrahedron. We apply the process again and get the union of the faces. We apply it once more and get the solid tetrahedron itself. We apply it again and still get just the solid tetrahedron. We need use the process just three times. We get a three-dimensional set, a 3-simplex.

If we had just one point, the application of the process would still leave us with just the one point. We need apply the process zero times. We get a zero-dimensional set, a 0-simplex. (We mention this case last because we have to understand the process before it can make much sense.)

Finally, let us consider a 3-simplex. Look at one of your models of tetrahedrons. It has four faces and each face is a 2-simplex. It has six edges and each edge is a 1-simplex. It has four vertices and each vertex is a 0-simplex.

Exercises 14-1b

1. (a) A 2-simplex has how many 1-simplexes as edges?

   (b) It has how many 0-simplexes as vertices?

2. A 1-simplex has how many 0-simplexes as vertices?

3. Using models show how two 3-simplexes can have an intersection which is exactly a vertex of each.

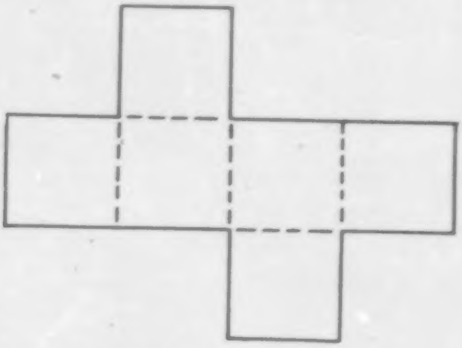
4. Using models show how two 3-simplexes can have an intersection which is exactly an edge of each.

5. In this and the next problem you are asked to do a bit of coloring. Mark three points not all on the same line in blue. Color red all points which are between any two of these. Shade green all points which are between any two of the points already colored. Should there be any points which are not colored and are between two of the colored points? Starting with the three points, how many times did you need to use the process of "taking points between" before you were finished?

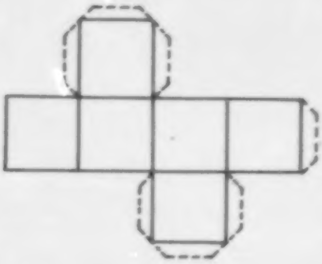
6. Use your model of a non-regular tetrahedron. Color its vertices blue. Color red the set of all points each of which is between two of the vertices. Color green the set of all points each of which is between two of the red or blue colored points. You should now have your model colored. What is the set of all points which either are colored or are between two of your colored points?

2. Polyhedrons.

Models of Cubes. Most of you know that if you want to make an ordinary box you need six rectangular faces for it. They have to fit and you have to put them together right. There is a rather easy way to make a model of a cube.



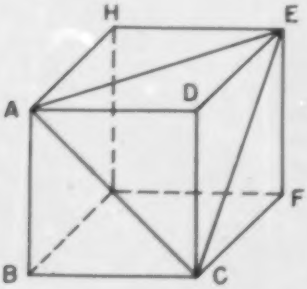
Draw six squares on heavy paper or cardboard as in the drawing above. Cut around the boundary of your figure and fold (or crease) along the dotted lines. Use cellulose tape or paste to fasten it together. If you are going to use paste it will be necessary to have flaps as indicated in the drawing below.

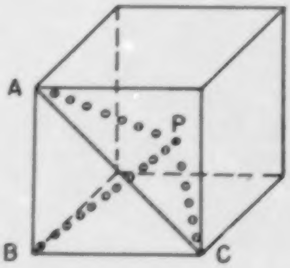


You will be asked to make two models of a cube in the exercises.

Can the surface of a cube be regarded as the union of 2-simplexes (that is, of triangles together with their interiors)? Can a solid cube be regarded as the union of 3-simplexes (that is of solid tetrahedrons)? The answer to both of these questions is "yes". We shall explain one way of thinking about these questions.

Each face of a cube can be considered to be the union of two 2-simplexes. The drawing on the left below shows a cube with three of its faces subdivided into two 2-simplexes each. The face ABCD appears as the union of (ABC) and (ACD) for example. The other faces which are indicated as subdivided are CDEF and ADEH. We can think of each of the other faces as the union of two 2-simplexes. Thus the surface of the cube can be thought of as the union of twelve 2-simplexes.





With the surface regarded as the union of 2-simplexes we may regard the solid cube as the union of 3-simplexes (solid tetrahedrons) as follows. Let P be any point in the interior of the cube. For any 2-simplex on the surface, (ABC), for example, (PABC) is a 3-simplex. In the figure on the right above, P is indicated as inside the cube. The 1-simplexes (PA), (PB), and (PC) would also be inside the cube. Thus with twelve 2-simplexes on the surface, we would have twelve 3-simplexes whose union would be the cube. The solid cube is the union of 3-simplexes in this nice way.

Now we ask another question. Do you suppose that a 3-simplex can be regarded as the union of a certain (finite) number of solid cubes? Can we find solid cubes that will fit together to fill up a 3-simplex? The answer to these questions is no. Suppose cubes could be fitted together to fill up a 3-simplex. Then any face of the 3-simplex would be filled up by square regions which are faces of the cubes. The square regions have right angles at their vertices. Any face of a 3-simplex is triangular. At least two of the angles of a triangle must be less than a right angle. Therefore the square regions cannot fit. A 3-simplex cannot be a finite union of cubes.

Exercises 14-2a

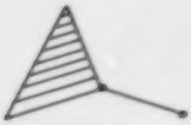
1. Make two models of cubes out of cardboard or heavy paper. Make them with each edge 2" long.
2. On one of your models, without adding any other vertices, draw segments to express the surface of the cube as a union of 2-simplexes. Label all the vertices on the model

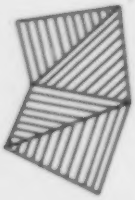
A, B, C, D, E, F, G, and H. Think of a point P in the interior of the cube. Using this point and the vertices of the 2-simplexes on the surface list the twelve 3-simplexes whose union is the solid cube.

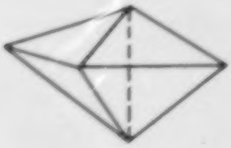
3. On the same cube as in problem 2, mark a point in the center of each face. (Each should be on one of the segments you drew in problem 2.) Draw segments to indicate the surface of the cube as the union of 2-simplexes using as vertices the vertices of the cube and these six new points you have marked. The surface is now expressed as the union of how many 2-simplexes?

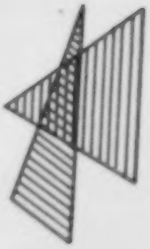
4. Think about a polyhedron formed by putting a square-based pyramid on each face of a cube. The surface of this new polyhedron has how many triangular faces? Can you compare this new polyhedron vertex for vertex, edge for edge, and 2-simplex for 2-simplex with the surface of the cube subdivided into 2-simplexes as in problem 3?

Polyhedrons. A polyhedron is the union of a finite number of simplexes. It could be just one simplex, or maybe the union of seven simplexes, or maybe of 7,000,000 simplexes. What we are saying is that it is the union of some particular number of simplexes. In the previous section, we observed that a solid cube, for example, was the union of twelve 3-simplexes. The figures below represent the unions of simplexes.





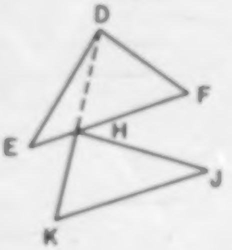


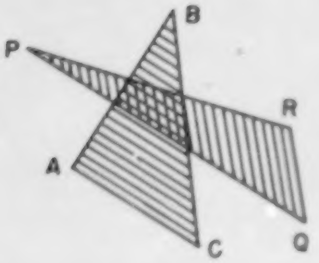


The figure on the left represents a union of a 1-simplex and a 2-simplex which does not contain the 1-simplex. It is therefore of mixed dimension. In what follows, we shall not be concerned with polyhedrons (or polyhedra) of mixed dimension. We assume a polyhedron is the union of simplexes of the same dimension. We shall speak of a 3-dimensional polyhedron as one which is the union of 3-simplexes. A 2-dimensional polyhedron is one which is the union of 2-simplexes. A 1-dimensional polyhedron is one which is the union of 1-simplexes. (Any finite set of points could be thought of as a 0-dimensional polyhedron but we won't be dealing with such here.)

The figure on the right above represents a polyhedron which seems to be the union of two 2-simplexes (triangular regions) but they don't intersect nicely. We prefer to think of a polyhedron as the union of simplexes which intersect nicely as in the middle two figures. Just what do we mean by simplexes intersecting nicely? There is an easy explanation for it. If two simplexes of the same dimension intersect nicely, then the intersection must be a face, or an edge, or a vertex of each. Mathematicians would say that they intersect "simplicially"; i.e., in a subsimplex of each.

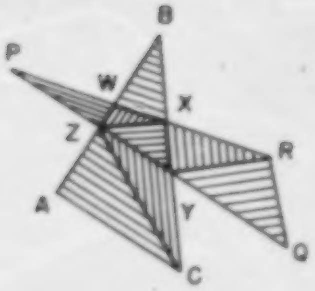
Let us look more closely at the union of simplexes which do not intersect nicely. In the figure on the right the 2-simplexes (DEF) and (HJK) have just the point H in common. They do not intersect nicely. While H is a vertex of (HJK), it is not of (DEF). However, the polyhedron which is the union of these two 2-simplexes is also the union of three 2-simplexes which do intersect nicely, namely, (DEH), (DHF), and (HJK).





The figure on the left represents the union of the 2-simplexes (ABC) and (PQR). They do not intersect nicely. Their intersection seems to be a quadrilateral together with its interior.

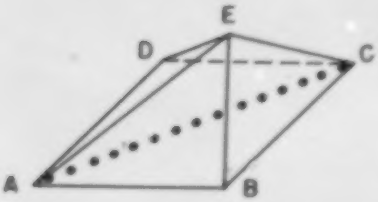
On the right we have indicated how the same set of points (the same polyhedron) can be considered to be a finite union of 2-simplexes which do intersect nicely. The polyhedron is the union of the eight 2-simplexes, (ACZ), (CZY), (PZW), (XYZ), (WXZ), (BWX), (XYR), and (YQR).

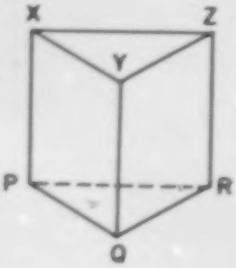


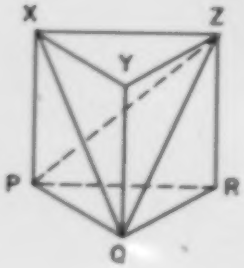
These examples suggest a fact about polyhedrons. If a polyhedron is the union of simplexes which intersect any way at all then the same set of points (the same polyhedron) is also the union of simplexes which intersect nicely. Except for the exercises at the end of this section, we shall always deal with unions of simplexes which intersect nicely. We will regard a polyhedron as having associated with it a particular set of simplexes which intersect nicely and whose union it is. When we say the word "polyhedron", we understand the simplexes to be there.

Is a solid cube a polyhedron, that is, is it a union of 3-simplexes? We have already seen that it is. Is a solid prism a polyhedron? Is a solid square-based pyramid? The answer to all of these questions is yes. In fact, any solid object each of whose faces is flat (that is, whose surface does not contain any curved portion) is a 3-dimensional polyhedron. It can be expressed as the union of 3-simplexes.

As examples let us look at a solid pyramid and a prism with a triangular base.







In the figure on the left the solid pyramid is the union of the two 3-simplexes (ABCE) and (ACDE). The figure in the middle represents a solid prism with a triangular base. The prism has three rectangular faces. Its bases are (PQR) and (XYZ). Here we see how it may be expressed as the union of eight 3-simplexes.

We use the same device we used for the solid cube. First we think about the surface as the union of 2-simplexes. We already have the bases as 2-simplexes. Then we think of each rectangular face as the union of two 2-simplexes. In the figure on the right above the face YZRQ is indicated as the union of (YZQ) and (QRZ), for instance. Now think about a point F in the interior of the prism. The 3-simplex (FQRZ) is one of eight 3-simplexes each with

with F as a vertex and whose union is the solid prism. In the exercises you will be asked to name the other seven.

Finally, how do we express a solid prism with a non-triangular base as a 3-dimensional polyhedron (that is, as a union of 3-simplexes with nice intersections)? We use a little trick. We first express the base as a union of 2-simplexes and therefore the solid prism as a union of triangular solid prisms. And we can then express each triangular solid prism as the union of eight 3-simplexes. We can do this in such a way that all the simplexes intersect nicely.

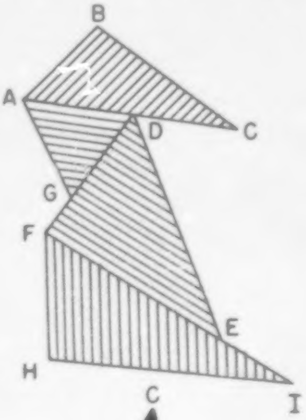
There is a moral to our story here. To do a harder-looking problem, we first try to break it up into a lot of easy problems each of which we already know how to do (or at least are able to do).

Exercises 14-2b

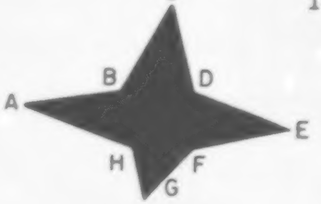
1. Draw two 2-simplexes whose intersection is one point and
   (a) is a vertex of each.
   (b) is a vertex of one but not of the other.
2. Draw three 2-simplexes which intersect nicely and whose union is itself a 2-simplex. (Hint: start with a 2-simplex as the union and subdivide it.)

3. You are asked to draw various 2-dimensional polyhedrons each as the union of six 2-simplexes. Draw one such that
   (a) No two of the 2-simplexes intersect.
   (b) There is one point common to all the 2-simplexes but no other point is common to any pair.
   (c) The polyhedron is a square together with its interior.

4. The figure on the right represents a polyhedron as the union of 2-simplexes without nice intersections. Draw a similar figure yourself and then draw in three segments to make the polyhedron the union of 2-simplexes which intersect nicely.

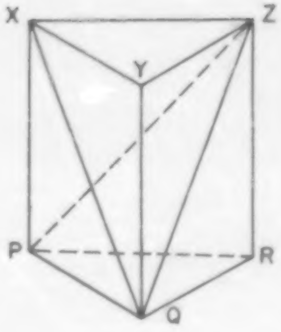


5. The 2-dimensional figure on the right can be expressed as a union of simplexes with nice intersections in many ways. Draw a similar figure yourself.



   (a) By drawing segments express it as the union of six 2-simplexes without using more vertices.
   (b) By adding one vertex near the middle (in another drawing of the figure), express the polyhedron as the union of eight 2-simplexes all having the point in the middle as one vertex.

6. (a) List eight 2-simplexes whose union is the surface of the triangular prism on the right. (The figure is like that used earlier.)



(b) Regarding F as a point in the interior of the prism list eight 3-simplexes (each containing F) whose union is the solid prism.

(c) The triangular prism PQRXYZ is also the union of *three* 3-simplexes which intersect nicely. Name such if you can.

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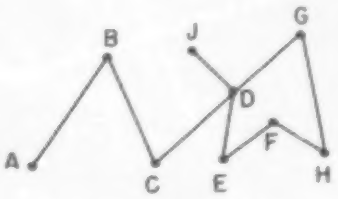
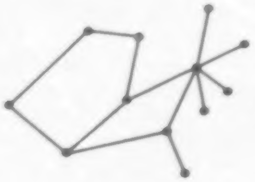
3. *Polyhedrons of Special Dimension.*

One-Dimensional Polyhedrons. A 1-dimensional polyhedron is the union of a certain number of 1-simplexes (segments). A 1-dimensional polyhedron may be contained in a plane or it may not be. Look at a model of a tetrahedron. The union of the edges is a 1-dimensional polyhedron. It is the union of six 1-simplexes and does not lie in a plane. We may think of the figures below as representing 1-dimensional polyhedrons that do lie in a plane (the plane of the page).

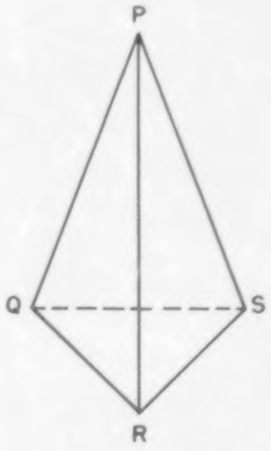




There are two types of 1-dimensional polyhedrons which are of special interest. A polygonal path is a 1-dimensional polyhedron in which the 1-simplexes can be considered to be arranged in order as follows. There is a first one and there is a last one. Each other 1-simplex of the polygonal path has one vertex in common with the 1-simplex which precedes it and one vertex in common with the 1-simplex which follows it. There are no extra intersections. The first and last vertices (points) of the polygonal path are called the endpoints.

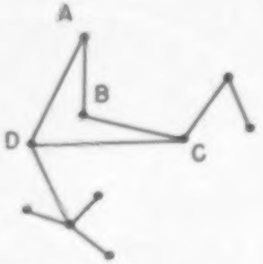
Neither of the 1-dimensional polyhedrons in the figures above is a polygonal path. But each contains many polygonal paths. The union of (AB), (BC), (CD), (DG) and (GH) is a polygonal path from A to H. The union of (JD) and (DE) is a polygonal path from J to E and consists of just two 1-simplexes.

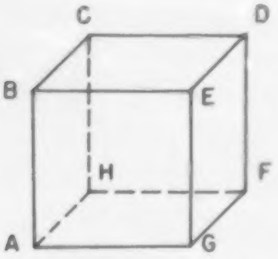
In the drawing on the right of a tetrahedron, the union of (PQ), (QR), and (RS) is a polygonal path from P to S (with endpoints P and S). The 1-simplex (PS) is itself a polygonal path from P to S. Consider the 1-dimensional polyhedron which is the union of the edges of the tetrahedron. Find three other polygonal paths from P to S in it. (Use a model if it helps you see it.)



The union of two polygonal paths that have exactly their endpoints in common is called a *simple closed polygon* (it is also a simple closed curve). Another way of describing a simple closed polygon is to say that is is a 1-dimensional polyhedron which is in one piece and has the property that every vertex of it is in exactly two 1-simplexes of it.

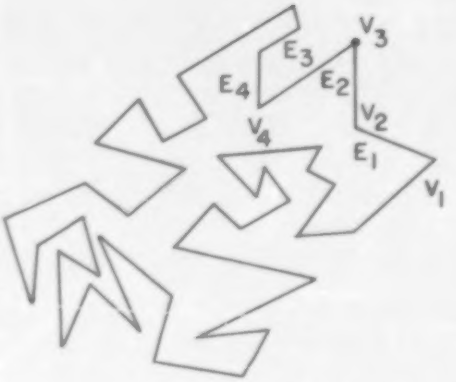
The 1-dimensional polyhedron on the right is not a simple closed polygon. But it contains exactly one simple closed polygon, namely the union of (AB), (BC), (CD), and (DA).





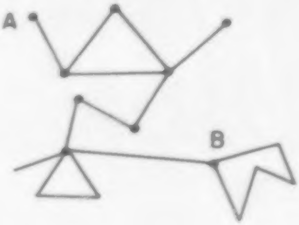
The union of the edges of the cube in the drawing on the left is a 1-dimensional polyhedron. It contains many simple closed polygons. One is the union of (AB), (BE), (EG), and (GA). Another is the union of (AB), (BC), (CD), (DE), (EG), and (GA). Can you give at least two more simple closed polygons containing (BE) and (GA)? (Use a model if it helps you see it.)

There is one very easy relationship on any simple closed polygon. The number of 1-simplexes (edges) is equal to the number of vertices. Consider the figure on the right. Suppose we start at some vertex. Then we take an edge containing this vertex. Next we take the other vertex contained in this edge and then the other edge containing this second vertex. We may think of numbering the vertices and edges as in the figure. We continue the process. We finish with the other edge which contains our original vertex. We start with a vertex and finish with an edge after having alternated vertices and edges as we go along. Thus the number of vertices is the same as the number of edges.

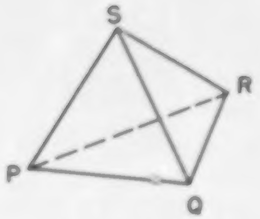


Exercises 14-3a

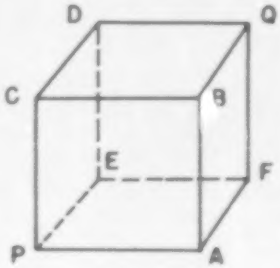
1. The figure on the right represents a 1-dimensional polyhedron. How many polygonal paths does it contain with endpoints A and B? How many simple closed polygons does it contain?



2. (a) The union of the edges of a 3-simplex (solid tetrahedron) contains how many simple closed polygons?
   (b) Name them all.
   (c) Name one that is not contained in a plane.
   (Use a model if you wish.)

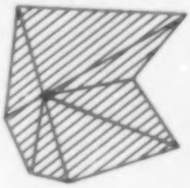


3. Let P and Q be vertices of a cube which are diametrically opposite each other (lower front left and upper back right). Name three polygonal paths from P to Q each of which contains all the vertices of the cube and is in the union of the edges. (Use a model if you wish.)

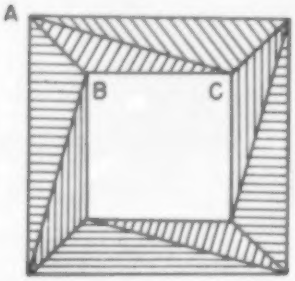


4. Draw a 1-dimensional polyhedron which is the union of seven 1-simplexes and contains no polygonal path consisting of more than two of these simplexes.
5. Draw a simple closed polygon on the surface of one of your models of a cube which intersects every face and which does not contain any of the vertices of the cube.

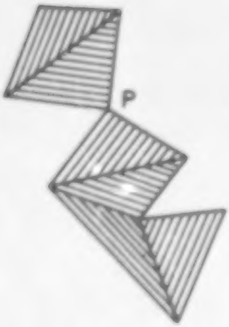
Two-Dimensional Polyhedrons. A 2-dimensional polyhedron is a union of 2-simplexes. As stated before, we agree that the 2-simplexes are to intersect nicely, that is, if two 2-simplexes intersect, then the intersection is either an edge of both, or a vertex of both. There are many 2-dimensional polyhedrons; some are in one plane but many are not in any one plane. The surface of a tetrahedron, for instance, is not in any one plane. Let us first consider a few 2-dimensional polyhedrons in a plane. In drawing 2-simplexes in a plane we shall shade their interiors.



Every 2-dimensional polyhedron in a plane has a boundary in that plane. The boundary is itself a 1-dimensional polyhedron. The boundary may be a simple closed polygon as in the figure on the right. In the figure on the left below we have indicated a polyhedron as the union of eight 2-simplexes. (ABC) is one of them.



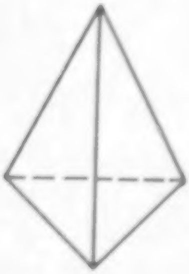
The boundary is the union of two simple closed polygons, the inner square and the outer square. These two polygons do not intersect.

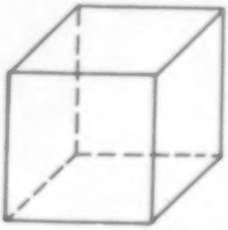


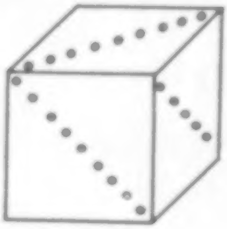
The figure on the right represents a 2-dimensional polyhedron which is the union of six 2-simplexes. The boundary of this polyhedron in the plane is the union of two simple closed polygons which have exactly one vertex of each in common, the point P.

Suppose a 2-dimensional polyhedron in the plane has a boundary which is a simple closed polygon (and nothing else). Then the number of 1-simplexes (edges) of the boundary is equal to the number of 0-simplexes (vertices) of the boundary. You have already seen, in the previous section, why this must be true.

There are many 2-dimensional polyhedrons which are not in any one plane. The surface of a tetrahedron is such a polyhedron; the surface of a cube is another (it may be considered to be expressed as a union of 2-simplexes). Here we have some 2-dimensional polyhedrons which are themselves surfaces or boundaries of 3-dimensional polyhedrons. Let us consider these two surfaces, the surface of a tetrahedron and the surface of a cube.







You may look at the drawings above or you may look at some models (or both). Let us count the number of vertices, the number of edges and the number of faces. But the surface of a cube can be considered in at least two different ways. We can think of the faces as being square regions (as in the middle figure) or we may think of each square face as subdivided into two 2-simplexes (as in the figure on the right). We will use F for the number of faces, E for the number of edges and V for the number of vertices. If you are counting from models and do not observe patterns to help you count, it is usually easier to check things off as you go along. That is, mark the objects as you count them.

Let us make up a table of our results.

	F	E	V
Surface of tetrahedron	?	6	?
Surface of cube (square faces)	?	?	8
Surface of cube (two 2-simplexes on each square face)	12	?	?

It is not easy from just these three examples to observe any nice relationship among these numbers. What we are looking for is a relationship which will be true not only for these 2-dimensional polyhedrons but also for others like these. Try and see if you can guess the relationship we will be telling you about in the last section.

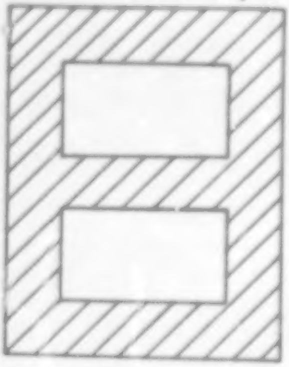
Exercises 14-3b

1. Make up a table as in the text showing F, V, and E for the 2-dimensional polyhedrons mentioned there.
2. Draw a 2-dimensional polyhedron in the plane with the polyhedron the union of ten 2-simplexes such that
   (a) its boundary is a simple closed polygon,
   (b) its boundary is the union of three simple closed polygons having exactly one point in common,
   (c) its boundary is the union of two simple closed polygons which do not intersect.
3. Draw a 2-dimensional polyhedron in the plane with the number of edges in the boundary
   (a) equal to the number of vertices,
   (b) one more than the number of vertices,
   (c) two more than the number of vertices.
4. Draw a 2-dimensional polyhedron which is the union of three 2-simplexes with each pair having exactly an edge in common. Do you think that there exists in the plane a polyhedron which is the union of four 2-simplexes such that each pair have exactly an edge in common?
5. On one of your models of a cube, mark six points one at the center of each face. Consider each face to be subdivided into four 2-simplexes each having the center point as a vertex.

Count F (the number of 2-simplexes), E (the number of 1-simplexes), and V (the number of 0-simplexes) for this subdivision of the whole surface. Keep your answers for later use.

6. Do the problem above without using a model and without doing any actual counting. Just figure out how many of each there must be. For instance, there must be 14 vertices, 8 original ones and 6 added ones.

7. Express the polyhedron on the right as a union of 2-simplexes which intersect nicely (in edges or vertices of each other).



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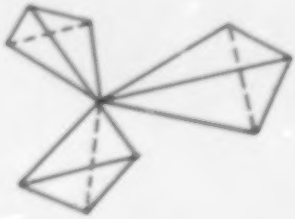
4. Three-Dimensional Polyhedrons, Simple Surfaces and the Euler Formula.

A 3-simplex is one 3-dimensional polyhedron. A solid cube is another 3-dimensional polyhedron. Any union of a certain number of 3-simplexes is a 3-dimensional polyhedron. We will assume again that the simplexes of a polyhedron intersect nicely. That is, that if two 3-simplexes intersect, the intersection is a 2-dimensional face (2-simplex) of each or an edge (1-simplex) of each or a vertex (0-simplex) of each.

Any 3-dimensional polyhedron has a surface (or boundary) in space. This surface is itself a 2-dimensional polyhedron. It is the union of several 2-simplexes (which intersect nicely). The surface of the 3-dimensional polyhedron represented by the drawing on the right is something of a mess. It consists of the surfaces of three tetrahedrons which have exactly one point in common.



The simplest kinds of surfaces of 3-dimensional polyhedrons are called simple surfaces. The surface of a cube and the surface of a 3-simplex are both simple surfaces. There are many others. Any surface of a 3-dimensional polyhedron obtained as follows will be a simple surface. Start with a solid tetrahedron. Then fasten another to it so that the intersection of the one you are adding with what you already have is a face of the one you are adding. You may keep adding more solid tetrahedrons in any combination or of any size provided that each one you add in turn intersects what you already have in a set which is exactly a union of one, two, or three faces of the 3-simplex you are adding. The surface of any polyhedron formed in this way will be a simple surface.

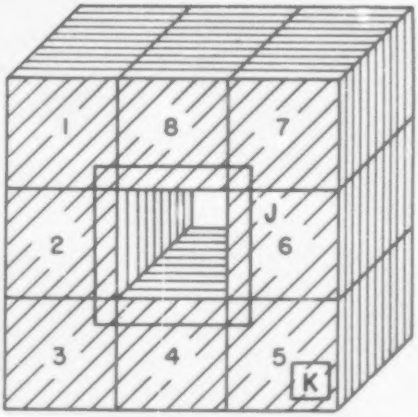
Group activity. Take five models of regular tetrahedrons of edges 3". Put marks on all four faces of one of these. Now fasten each of the others in turn to one of the marked faces. The marked one should be in the middle and you won't see it any more. The surface of the object you have represents a simple surface. You can see how to fasten a few more tetrahedrons on to get more and more peculiar looking objects. Suppose it is true that whenever you add a solid tetrahedron the intersection of what you add with what you already have is one face, two faces or three faces of the one you add. The surface of what you get will be a simple surface.

One can also fasten solid cubes together to get various 3-dimensional polyhedrons. In fastening solid cubes in turn onto what you already have, you will always wind up with a 3-dimensional polyhedron which has a simple surface provided the following condition is met. The intersection of each one you add in turn with what you already have must be a set *which is bounded on the surface of the cube you are adding by a simple closed polygon*. For example, the intersection might be a face or the union of two adjacent faces of the one you add.

Finally we mention an interesting property of simple surfaces. Draw any simple closed polygon on a simple surface. Then this polygon separates the simple surface into exactly two sets each of which is connected; i.e., is one piece.

Group activity. On the surface of one of the peculiar 3-dimensional polyhedrons (with simple surface) that you have constructed above, have somebody draw any simple closed polygon (the wilder the better). It need not be in just one face. Then have somebody else start coloring somewhere on the surface but away from the polygon. Have him color as much as he can *without crossing the polygon*. Then have another person start coloring with another color at any previously uncolored place. Color as much as possible but *do not cross the polygon*. When the second person has colored as much as possible, the whole surface should be colored.

If you don't carefully follow the instructions for constructing a polyhedron with simple surface you may get a polyhedron whose surface is not simple. Suppose, for instance, you fasten eight cubes together as in the drawing below. The polyhedron looks something like a square doughnut. Note that in fitting the eighth one, the intersection of the one you are adding with what you already have is the union of two faces which are *not* adjacent. The boundary (on the eighth cube) of the intersection is two simple closed polygons, not just one as it should be. There are many simple closed polygons on this surface which do not separate it at all. The polygon J does not separate it. The polygon K does.

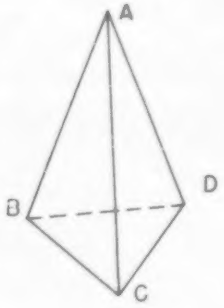


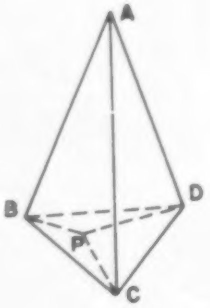
Exercises 14-4a

1. Using a block of wood (with corners sawed off if possible), draw a simple closed polygon on the surface making it intersect most or all of the faces of the solid. Start coloring at some point. Do not cross the polygon. Color as much as you can without crossing the polygon. When you have colored as much as you can, start coloring with a different color on some uncolored portion. Again color as much as you can without crossing the polygon. You should have the whole surface colored when you finish.
2. Go through the same procedure as in problem 1 but with another 3-dimensional solid. Use one of your models or another block of wood. Make your simple closed polygon as complicated as you wish.

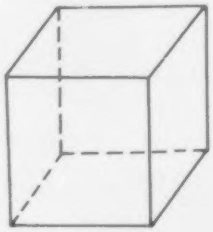
Counting Vertices, Edges, and Faces--the Euler Formula.

In Section 14-3 you were asked to do some counting. A few of you may have discovered a relationship between F, E, and V. Consider the tetrahedron in the figure below. Its surface is a simple surface. What relationship can we find among the vertices, edges, and faces of it?





There are the same number of edges and faces coming into the point A, three of each. One may see that on the base there are the same number of vertices and edges. We have two objects left over: the vertex A at the top and the face (BCD) at the bottom. Otherwise we have matched all the edges with vertices and faces. So $F + V - E = 2$. Now let us ask what would be the relationship if one of the faces or the base were broken up into several 2-simplexes. Suppose we had the base broken up into three 2-simplexes by adding one vertex P in the base. The figure on the right above illustrates this. Our counting would be the same until we got to the base and we would be able to match the three new 1-simplexes which contain P with the three new 2-simplexes on the base. We have lost the face which is the base but we have picked up one new vertex P. Thus the number of vertices plus the number of 2-simplexes is again two more than the number of 1-simplexes. F + V - E = 2.



Now let us look at a cube. We have a drawing of one on the right. The cube has how many faces? How many edges? How many vertices? Is the sum of the number of vertices and the number of faces two more than the number of edges? Let us see why this must be.

(1) The number of vertices on the top face is the number of edges on the top face.

(2) The number of vertices on the bottom face is the number of edges on the bottom face.

(3) The number of vertical faces is the number of vertical edges.

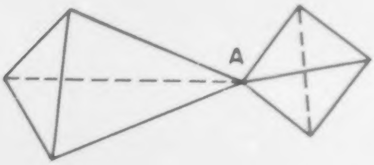
(4) All the vertices and edges are now used up. All the vertical faces are now used up. We have the top and bottom faces left.

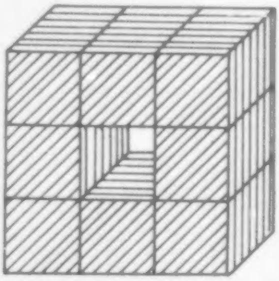
So F + V - E must be 2.

What would happen if each face were broken up into two 2-simplexes? For each face of the cube you would now have two 2-simplexes. But for each face you would have one new 1-simplex lying in it. Other things are not changed. Hence F + V - E is again 2.

Suppose we have any simple surface. Then do you suppose that V + F - E = 2? In the exercises you will be asked to verify this formula (which is known as the Euler Formula) in several other examples. (Euler--pronounced "oiler"--was the name of a famous mathematician of the early 18th century.)

Let us now observe that the formula does not hold in general for surfaces which are not simple. Consider the two examples below.





In the figure on the left (the union of the two tetrahedrons having exactly the vertex A in common) V + F - E = ? Count and see. Use models of two tetrahedrons if you wish. V + F - E should be 3. On each tetrahedron separately the number of faces plus the number of vertices minus the number of edges is 2. But the vertex A would have been counted twice. So V + F is one less than E + 4.

The figure on the right above is supposed to represent the union of eight solid cubes as in the last section. The possible ninth one in the center is missing. Count all the faces (of cubes), edges and vertices which are in the surface. For this figure V + F - E should be 0. (As a starter, V should be 32.)

Finally we put the Euler Formula in a more general setting. Suppose we have a simple surface and it is subdivided into a number (at least three) of non-overlapping pieces. Each of these pieces is to be bounded on the surface by a simple closed polygon. We think of F as the number of these pieces. We require that if two of these pieces intersect then the intersection be either one point or a polygonal path. The number E is the number of these intersections of pairs of pieces which are not just points. The number V is the number of points each of which is contained in at least three of these pieces. Then F + V - E = 2.

Exercises 14-4b

1. Take a cardboard model of a non-regular tetrahedron. In each face add a vertex near the middle. Consider the face as the union of three 2-simplexes so formed. Give the count of the faces, edges, and vertices of the 2-simplexes on the surface. How do the faces, edges, and vertices of this polyhedron compare with those of the polyhedron you get by attaching four regular tetrahedrons to the four faces of a fifth?

2. Take a model of a cube. Subdivide it as follows. Add one vertex in the middle of each edge. Add one vertex in the middle of each face. Join the new vertex in the middle of each face with the eight other vertices now on that face. You should have eight 2-simplexes on each face. Compute F, V and E. Do you get F + V - E = 2?
3. Make an irregular subdivision of any simple surface into a number of flat pieces. Each piece should have a simple closed polygon as its boundary. Count F, V, and E for this subdivision of the surface.